Systemic Risk Driven Portfolio Selection

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Abstract

We consider an investor who maximizes portfolio's expected returns conditioned on the occurrence of a systemic event: financial system return being at, or at most at, its VaR level and portfolio's returns being below the CoVaR level. We obtain a closed-form solution to the portfolio selection problem, and show how VaR and CoVaR quantiles control, respectively, the relative importance of "portfolio–system correlation" and "portfolio variance". Our empirical analysis demonstrates that the investor attains a higher Sharpe ratio, compared to well known benchmark portfolio criteria, during times of market downturn. Portfolios that perform best in adverse market conditions are less diversified and concentrate on few stocks whose correlation with the financial system is low.

Keywords: Systemic risk, Portfolio selection, Risk management, Sharpe ratios

JEL classification: G01, G11, G20, G28

1 Introduction

The balance between risk and return has been at the center of portfolio construction since the seminal work of Markowitz (1952). The asset allocation literature has primarily focused on a firm's individual risk. The global 2007–2009 financial crisis has highlighted the importance of accounting for systemic events, i.e., extreme form of risks that can have severe consequences on the financial

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system. Important studies that identify and measure systemic risk of financial institutions include Adrian and Brunnermeier (2016), Brownlees and Engle (2017), and Acharya et al. (2012). The central question we study in this paper is: how to construct portfolios that perform well in the face of systemic events?

We consider portfolio returns conditioned on a systemic event which corresponds to the realization of an extremely adverse market outcome. The investor makes his portfolio decisions taking into account the state of the financial system ("system" hereafter) defined as the portfolio of all financial institutions. We measure the tail risk using two measures: VaR and CoVaR. The VaR of the system is defined as the most adverse change of system returns at some pre-specified level of confidence, while CoVaR is the VaR of the portfolio conditioned on the system being at (or below) its VaR level. The goal of our investor is to maximize the expected portfolio returns conditioned on (i) the system being at (or below) its VaR level, *and* (ii) the portfolio returns being below their CoVaR level. In other words, we seek the portfolio that performs best in a low return environment and when the system is in distress. In the portfolio optimization problem, we consider both the original CoVar formulation proposed by Adrian and Brunnermeier (2016), which conditions on the system being *at* its VaR level, and the modified version of Girardi and Ergun (2013), which conditions on the event that the system is *at most at* its VaR level. We refer to the former as the optimistic case, and to the latter as the pessimistic case. This is because in the latter approach the investor considers more extreme scenarios of systemic events, that is, when the system returns are worse than their VaR level.

There are several contributions in our efforts. First, under the joint normality assumption on portfolio and system returns we obtain closed-form expressions for the investor's portfolio problem. In the pessimistic case, we show that our investor solves an optimization problem that resembles a traditional mean–variance optimization, with the important difference that the risk-aversion parameter is not constant but rather depends nonlinearly on the correlation between the investor's portfolio and the system. Our solution technique for solving the optimization problem relies on a Taylor series expansion that converts the non-linear optimization problem into a set of simpler problems, and a fixed point iteration to determine which problem from the set admits a solution that coincides with that of the original optimization problem (see Section 2.2 for the details). We establish the following mutual fund separation result: any optimal portfolio with a given expected

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return, variance, and covariance with the system can be replicated by only *three* appropriately chosen optimal portfolios. In addition, we show that when the assets in the portfolio are uncorrelated with the system, the optimal portfolios are mean–variance efficient. Second, we provide an economic explanation for the main drivers behind the investor's portfolio selection. We show that the VaR and CoVaR quantile parameters determine the weights assigned to portfolio–system correlation and portfolio variance, respectively, in the investor's objective function. We prove that when the standard deviation of the portfolio increases, portfolio returns become more sensitive to the correlation of the portfolio with the system. This result can be intuitively understood as follows: when the variance of the portfolio is large, the downside risk is large if the portfolio is highly correlated with the system (portfolio returns are bounded above by CoVaR and negative portfolio returns become more likely).

We assess the performance of the proposed methodology on the Canadian and US stock markets. We choose the constituents of the S&P500 Financials Index as the portfolio components in the US market, and the constituents of the S&P/TSX Capped Financial Index as those for the Canadian market. We consider the time period from January 4, 2000 until October 1, 2018, hence covering the global 2007-2009 financial crisis when noticeable systemic events occurred. We use the GARCH Dynamic Conditional Correlation (GARCH–DCC) model (see Engle (2002, 2009)) to model the joint return dynamics of the stocks and the system, which we proxy by the MSCI World Index. We compare our portfolio criteria with two well known benchmarks, the global Minimum Variance (MV) and $1/n$ portfolios (see DeMiguel et al. (2009)).[1] The out-of-sample analysis of portfolios' Sharpe ratios reveals that our optimal portfolio outperforms these benchmarks at times of market downturns. We also find that the optimal portfolio is less diversified than the MV and $1/n$ portfolios. These results can be intuitively explained as follows: portfolios that behave well during crisis periods tend to invest on few stocks that have relatively small correlation with the system.[2]

Our paper contributes to a rather scarce literature on (C)VaR-based portfolio selection. Alexander and Baptista (2002) consider the problem of minimizing an investor's criterion that trades off

[1]The global Minimum Variance (MV) portfolio is the portfolio that one obtains by minimizing portfolio variance without any restrictions on portfolio expected return (the portfolio with the smallest variance in the Markowitz portfolio selection problem). The $1/n$ portfolio is the strategy that invests equally on each stock of the portfolio.

[2]These findings are reminescent of those in the literature on systemic risk in financial networks (see for example, Acemoglu et al. (2015) and Capponi el al. (2016)). They show that a higher concentration of interbank liabilities may be socially preferable to a well diversified interbanking network if the financial system experiences a large shock.

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expected return with value at risk, and characterize the mean-VaR efficient set. Alexander and Baptista (2004) extend their earlier work to analyze the conditions under which a CVaR constraint is more effective than the VaR constraint as a risk management tool. Rockafellar and Uryasev (2000) study the optimal portfolio selection problem in which the CVaR is the risk minimization criterion. Biglova et al. (2014) study a portfolio selection problem that accounts for systemic risk by optimizing the average portfolio loss when all assets in the portfolio are distressed, i.e., below their individual VaR levels. Unlike Biglova et al. (2014), we obtain a closed-form solution for the portfolio problem and exploit its representation to obtain insights on the role played by the VaR and CoVaR quantile parameters. None of the above surveyed papers explicitly incorporates systemic risk in portfolio selection and (C)VaR is evaluated only for the portfolio, without accounting for the state of the financial system. In this respect, it is important to recognize that poor performance of a given portfolio (which could be industry specific) does not necessarily imply poor state of the broad economy.

The rest of the paper is organized as follows. In Section 2 we solve the portfolio selection problem: when the system is assumed to be at its VaR level (Section 2.1) and when the system is at most at its VaR level (Section 2.2). In Section 3 we discuss the relation of our framework to the mean–variance analysis. In Section 4, we assess the empirical performance of our model on equity data from the Canadian and the US market. Section 5 concludes. Proofs of technical results are deferred to the appendix.

2 Portfolio Selection Under Systemic Risk

We formulate and study the optimization problem of an investor, who accounts for systemic events in his portfolio construction. In Section 2.1, we consider the situation in which the systemic event occurs when the financial system is at its VaR level (optimistic case). In Section 2.2, we analyze the situation in which the systemic event occurs when the financial system is below its VaR level (pessimistic case). We derive the optimal asset allocation for an investor who maximizes the expected returns on his portfolio, conditional on the systemic event and on the portfolio returns being below their CoVaR levels. We assume that there exists no risk-free asset, and there are $n \geq 2$ risky assets with stochastic rates of return $r = (r_1, ..., r_n)^{\mathsf{T}}$. Before proceeding further with the

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analysis, we introduce notation used throughout the paper.

Notation. The vector of expected rates of return is denoted by $\mu = \mathbb{E}[r]$, and we use $\Sigma = \mathbb{E}\left[(r-\mu)(r-\mu)^{\mathsf{T}}\right]$ to denote the variance–covariance matrix of the rates of return. We use $w = (w_1, ..., w_n)^{\mathsf{T}}$ to denote the vector of weights, i.e., w_i is the proportion of wealth invested in asset i, thus implying that $\sum_{i=1}^{n} w_i = 1$. Let $R_p = w^{\mathsf{T}} r$ and $\mu_p = w^{\mathsf{T}} \mu$ denote, respectively, the portfolio's rate of return and portfolio's expected rate of return. Then the variance of portfolio returns is given by $\sigma_p^2 = w^{\mathsf{T}} \Sigma w$. We use R_m to denote the return on the system. We denote by μ_m and σ_m, respectively, the expected return and standard deviation of the return on the system. We use σ to denote the column vector of covariances of each asset with the system. $\phi(\cdot)$ denotes the probability density function (pdf) of a standard Gaussian, and $\Phi(\cdot)$ the cumulative distribution function (cdf) of a standard Gaussian. $\mathbf{0}$ and $\mathbf{1}$ are column vectors of zeros and ones, respectively, whose dimension is understood from the context.

2.1 System is at its VaR level

The system's VaR is defined as the value VaR_{q_m} such that

$$\mathbb{P}\left(R_m \leq VaR_{q_m}\right) = q_m, \tag{2.1}$$

where q_m is the quantile level. The lower q_m, the smaller the value of VaR_{q_m}. Following Adrian and Brunnermeier (2016), we define the CoVaR of a portfolio, denoted by $CoVaR_{q_p}$, as

$$\mathbb{P}\left(R_p \leq CoVaR_{q_p} \middle| R_m = VaR_{q_m}\right) = q_p, \tag{2.2}$$

where R_p is the return on the portfolio and q_p the quantile level. The lower q_p, the smaller the value of $CoVaR_{q_p}$. In plain words, CoVaR is defined as the VaR of the portfolio conditional on the financial system being in distress. Hence, CoVaR addresses the question of *what portfolios are most exposed to a financial crisis.*

Assumption 1 *Because our focus is on stressed state of the economy, we assume throughout the paper that $q_m, q_p < 0.5$, i.e., the portfolio and system returns are below their median value.*

From the investor's perspective, it is desirable to construct a portfolio that performs well when

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the entire financial system is in a downturn. To immunize a given portfolio against market downturns, we incorporate systemic risk directly in the portfolio optimization procedure. The performance criterion of the investor is the co-expected returns defined as

$$CoER^{=} = \mathbb{E}\left[R_p \middle| R_p \leq CoVaR_{q_p}, R_m = VaR_{q_m}\right]. \tag{2.3}$$

We emphasize that in (2.3) we condition on the event of low portfolio returns ($\{R_p \leq CoVaR_{q_p}\}$) and stressed market conditions ($\{R_m = VaR_{q_m}\}$). We graphically illustrate the portfolio returns under the specification (2.3) in Figure 1.

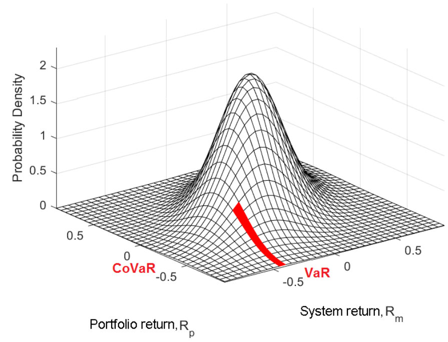

Figure 1: Returns on which the conditioning used in the definition of $CoER^{=}$ is applied. For a given probability distribution of returns on the system and portfolio (grey), we consider only those returns that correspond to stressed scenarios (red): portfolio returns are below CoVaR ($R_p \leq CoVaR_{q_p}$) and the system returns are at their VaR level ($R_m = VaR_{q_m}$).

In words, $CoER^{=}$ estimates the expected returns in a low return environment when the overall system is in distress (system is *at* its VaR level). Thus, the portfolio selection problem can be stated as[3]

$$\begin{aligned} \max_{w} \quad & CoER^{=} \\ \text{s.t.} \quad & w^{\mathsf{T}}\mathbf{1} = 1, \end{aligned} \tag{\textbf{P1}}$$

By solving (**P1**) we find the portfolio that performs well when the system is at its VaR level and portfolio's returns are below CoVaR.

[3]One can additionally impose a constraint on the *unconditional* expected portfolio returns, that is, enlarge the set of constraints in (**P1**) with $\mu^{\mathsf{T}}\mathbf{1} = \mu_p$. The problem can still be solved in closed form.

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To solve the portfolio selection problem (**P1**), we first obtain a closed-form expression for $CoER^{=}$. Using such a representation, we show that the quantile parameters q_p and q_m determine, respectively, the weights that the investor assigns to the portfolio variance and portfolio–system correlation. We make the following assumption throughout the paper:

Assumption 2 *The joint portfolio and system returns $(R_p, R_m) \sim \mathcal{BN}(\mu_p, \mu_m, \sigma_p, \sigma_m, \rho)$, where $\mathcal{BN}$ denotes the bivariate Gaussian distribution, μ_p and μ_m are, respectively, the expected portfolio and system return, σ_p and σ_m are, respectively, the standard deviation of the portfolio and system return, and ρ is the correlation between portfolio and system returns.*

Aside from analytical tractability, the above assumption allows us to obtain clear economic intuitions for the results. Notably, we obtain a closed-form solution for the optimal portfolio, and analytically compare it with well-known portfolio selection techniques, such as the mean–variance and the $1/n$ portfolio, that do not account for systemic risk.

Proposition 1 *Fix a vector w of weights of each asset in the portfolio. Then, under Assumption 2 we have*

$$CoER^{=} = \underbrace{w^{\top}\mu}_{\substack{\textit{Portfolio} \\ \textit{Return}}} + \frac{\Phi^{-1}(q_m)}{\sigma_m} \underbrace{w^{\top}\sigma}_{\substack{\textit{Portfolio-System} \\ \textit{Covariance}}} - \frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p} \sqrt{\underbrace{w^{\top}\Sigma w}_{\substack{\textit{Portfolio} \\ \textit{Variance}}} - \frac{1}{\sigma_m^2}\left(w^{\top}\sigma\right)^2}, \tag{2.4}$$

where q_m and q_p are the quantile levels used in the definition of VaR and CoVaR, respectively (see (2.1) *and* (2.2)*), and σ is the column vector of covariances of each risky asset with the system. Moreover, the following holds*

$$\frac{\mathrm{d}CoVaR_{q_p}}{\mathrm{d}\rho} < 0 \textit{ if and only if } \frac{\rho}{\sqrt{1-\rho^2}} < \frac{\Phi^{-1}(q_m)}{\Phi^{-1}(q_p)}. \tag{2.5}$$

Notice that $CoER^{=}$ only depends on the first two moments, which follows from the fact that joint returns are Gaussian. $CoER^{=}$ consists of three components: expected portfolio returns, portfolio–system covariance, and portfolio variance.

The parameters q_m and q_p allow the investor to balance the significance of these three components, relative to one another, in the portfolio's $CoER^{=}$. The parameter q_m controls the weight

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that the investor assigns to the covariance between portfolio and system returns when evaluating portfolio's $CoER^{=}$. Small q_m implies a large weight (i.e., large $\Phi^{-1}(q_m)$ in absolute terms) given to the portfolio's covariance with the system: if q_m is small, the investor considers more adverse scenarios of system returns and, consequently, he dislikes a portfolio whose returns have high positive correlation with the system. Since in most practically relevant cases $q_m < 0.5$, and thus $\Phi^{-1}(q_m) < 0$, investors will prefer portfolios that are negatively correlated with the system because, *ceteris paribus*, such portfolios have larger $CoER^{=}$. The parameter q_p controls the weight that the investor assigns to portfolio variance. It follows from (2.4) that the investor is able to define relative significance of the variance only to a certain extent as the weight $\frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}$ is multiplying the difference between portfolio variance and squared portfolio–system covariance (divided by σ_m^2), not the portfolio variance *per se*. Nonetheless, we can interpret $\frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}$ as the weight that defines the relative significance of the variance, if we fix the covariance of the portfolio with the system. If q_p is small, the investor assigns a large weight to the portfolio variance (see the Appendix for the proof of the inequality $\frac{\mathrm{d}}{\mathrm{d}q_p}\left(\frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}\right) < 0$), implying that he is less tolerant against adverse portfolio returns: because these returns are bounded from above by $CoVaR_{q_p}$, large portfolio variance implies large downside risk in the sense that low portfolio returns become more likely. The fact that $CoVaR_{q_p}$ can increase with the correlation parameter $\rho > 0$ may be explained as follows. The value of $CoVaR_{q_p}$ depends on mean and variance of conditional portfolio returns (see Eq.(A.3) in the Appendix)

$$R_p|R_m = VaR_{q_m} \sim \mathcal{N}\left(\mu_p + \rho\sigma_p\Phi^{-1}(q_m), \sigma_p^2(1-\rho^2)\right).$$

Clearly, a decrease in mean implies a *decrease* in $CoVaR_{q_p}$, whereas a decrease in variance implies an *increase* in $CoVaR_{q_p}$ (see (2.2)). Since both the mean and the variance of conditional returns $R_p|R_m = VaR_{q_m}$ decrease with the correlation parameter ρ, the sensitivity of $CoVaR_{q_p}$ to ρ depends on which force dominates. On the one hand, when ρ is low the sensitivity of the mean to an increase in ρ is higher, and $CoVaR_{q_p}$ decreases. On the other hand, if ρ is high the sensitivity of variance to an increase in ρ is higher, and $CoVaR_{q_p}$ increases. If $\rho \approx 1$, we expect the variance effect to dominate because the portfolio return R_p becomes very close to the constant system return $R_m = VaR_{q_m}$ (the conditioning event is $R_m = VaR_{q_m}$). In financial terms, as ρ approaches one,

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the uncertainty regarding portfolio return R_p vanishes.

The optimal portfolio that maximizes $CoER^{=}$ is given in the following proposition.

Proposition 2 *Under Assumption 2, the portfolio that maximizes $CoER^{=}$ is given by*

$$w^* = \frac{\widehat{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\mathsf{T}\widehat{\Sigma}^{-1}\mathbf{1}} - \frac{1}{\sqrt{\left(\lambda^2 - \Delta^\mathsf{T} Q^{-1}\Delta\right)\left(\mathbf{1}^\mathsf{T}\widehat{\Sigma}^{-1}\mathbf{1}\right)}} \begin{bmatrix} (Q^{-1})^\mathsf{T}\Delta \\ -\mathbf{1}^\mathsf{T} Q^{-1}\Delta \end{bmatrix}$$

provided that

$$\lambda > \sqrt{\Delta^\mathsf{T} Q^{-1}\Delta}, \tag{2.6}$$

where

$$\lambda = \frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}, \quad \widehat{\Sigma} = \Sigma - \frac{1}{\sigma_m^2}\sigma\sigma^\mathsf{T}.$$

and the expressions of Q and Δ are given by (A.7) *in the Appendix.*

The technical condition (2.6) guarantees that the portfolio w^* maximizing $CoER^{=}$ is bounded. If this condition is violated, it becomes possible to construct a portfolio with infinitely high expected return, as illustrated in the following example.

Example 1 *Consider a portfolio consisting of two assets, and assume that expected returns, the covariance of the assets with the system, and the correlation matrix are given by*

$$\mu = \begin{array}{cc} \textit{Asset 1} & \textit{Asset 2} \\ (0.280 & 0.080)^\mathsf{T}, \end{array} \qquad \sigma = \begin{array}{cc} \textit{Asset 1} & \textit{Asset 2} \\ (0.035 & 0.029)^\mathsf{T}, \end{array} \qquad \begin{array}{ccc} \textit{Asset 1} & \textit{Asset 2} & \textit{System} \end{array} \\ \begin{pmatrix} 1.000 & 0.550 & 0.750 \\ 0.550 & 1.000 & 0.660 \\ 0.750 & 0.660 & 1.000 \end{pmatrix}.$$

Assume that the variances of the assets and of the system are respectively given by $\sigma_1 = 0.036, \sigma_2 =$ 0.033, and $\sigma_m^2 = 0.059$. The portfolio that maximizes $CoER^{=}$ is then

$$w^* = \begin{array}{cc} \textit{Asset 1} & \textit{Asset 2} \\ (0.748 & 0.252)^\mathsf{T}. \end{array}$$

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If the correlation of the asset 1 with the system were to decrease from 0.750 *to* 0.100*, then condition* (2.6) *is no longer satisfied, and the investor finds it optimal to hold a very large long position in the first asset. This investment strategy allows him to simultaneously increase portfolio expected returns and decrease the portfolio correlation with the system, a desirable outcome in periods of market downturns.*[4]

2.2 System is at most at its VaR Level

In this section, we account for more severe systemic events when selecting the optimal portfolio. Such a higher severity is captured through a generalization of the CoVaR measure proposed by Girardi and Ergün (2013). In this modified CoVaR, the conditioning event is a more extreme market downturn. We condition on $R_m \leq VaR_{q_m}$ rather than $R_m = VaR_{q_m}$, that is,

$$\mathbb{P}\left(R_p \leq CoVaR_{q_p} \middle| R_m \leq VaR_{q_m}\right) = q_p. \tag{2.7}$$

Hence, losses that are farther out in the tail of the distribution, i.e. those beyond VaR_{q_m}, are accounted for. We define the co-expected return as

$$CoER^{\leq} = \mathbb{E}\left[R_p \middle| R_p \leq CoVaR_{q_p}, R_m \leq VaR_{q_m}\right],$$

where $CoVaR_{q_p}$ is defined by Eq. (2.7). Figure 2 graphically illustrates system returns upon which we condition.

Similarly to $CoES^{=}$, $CoER^{\leq}$ estimates the expected returns in a low return environment when the system is in distress (system is at most at its VaR level). Thus, the portfolio selection problem can be stated as

$$\begin{aligned} \max_{w} \quad & CoER^{\leq} \\ \text{s.t.} \quad & w^{\intercal}\mathbf{1} = 1. \end{aligned} \tag{\textbf{P2}}$$

In other words, the optimal portfolio that solves (**P2**) is expected to perform well when the system

[4] To guarantee that Assumption (2.6) is satisfied, one can impose additional constraints in problem (**P1**) such as limiting the amount of short-selling allowed.

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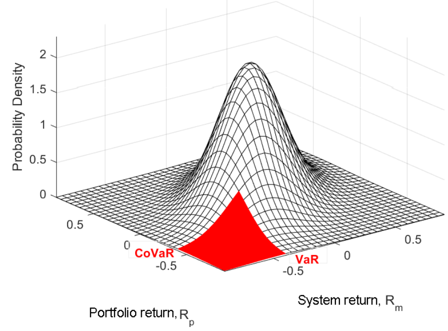

Figure 2: Returns on which the conditioning event used in the definition of $CoER^{\leq}$ is applied. For a given probability distribution of returns on the system and portfolio (in grey), we consider only those returns that correspond to stressed scenarios (in red): portfolio returns are below CoVaR ($R_p \leq CoVaR_{q_p}$) and the system returns are at most at their VaR level ($R_m \leq VaR_{q_m}$).

is at most at its VaR level and portfolio's returns are below CoVaR. The following proposition derives a closed-form expression for $CoER^{\leq}$.

Proposition 3 *Under Assumption 2, we have that*[5]

$$CoER^{\leq} = \mu_p - \overline{\lambda}(\rho; q_m, q_p)\sigma_p, \tag{2.8}$$

where $\overline{\lambda}(\rho; q_m, q_p)$ *is given by*

$$\overline{\lambda}(\rho; q_m, q_p) = \frac{1}{q_m q_p}\left(\phi(\eta_1)\Phi\left(\frac{\eta_2 - \rho\eta_1}{\sqrt{1-\rho^2}}\right) + \rho\phi(\eta_2)\Phi\left(\frac{\eta_1 - \rho\eta_2}{\sqrt{1-\rho^2}}\right)\right). \tag{2.9}$$

Above, we have defined $\eta_1 := \frac{CoVaR_{q_p} - \mu_p}{\sigma_p}$, *and* $\eta_2 := \frac{VaR_{q_m} - \mu_m}{\sigma_m}$. *In addition, it holds*

$$\frac{\mathrm{d}CoVaR_{q_p}}{\mathrm{d}\rho} < 0, \quad \frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho} = \frac{1}{q_m q_p}\phi(\eta_2)\Phi\left(\frac{\eta_1 - \rho\eta_2}{\sqrt{1-\rho^2}}\right) > 0. \tag{2.10}$$

We highlight here the important properties of $CoER^{\leq}$. The function $\overline{\lambda}$ can be regarded as the weight that the investor assigns to portfolio standard deviation σ_p – the measure of riskiness of the portfolio. It then follows from (2.10) that when systemic risk is accounted for, this weight increases with the portfolio–system correlation ρ ($\frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho} > 0$). This can be explained as follows. Since by

[5] To make the notation less cumbersome, we do no explicitly highlight the dependence of the correlation ρ on the portfolio weights w. However, it should be understood that $\rho = \rho(w)$.

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construction of $CoER^{\leq}$ the system is assumed to be stressed, higher correlation with the system implies larger potential losses for the portfolio whose downside is unbounded, but the upside is bounded by $CoVaR_{q_p}$. This in turn implies that standard deviation is especially important for the portfolio selection.

Compared with $CoER^{=}$, the dependence of the $CoER^{\leq}$ on the quantile parameters q_p and q_m has a more complex form. To visually explain such a dependence structure, we plot the contour lines of $CoER^{\leq}$ in Figure 3.[6] First, it clearly appears from panels (a) and (b) of Figure 3 that,

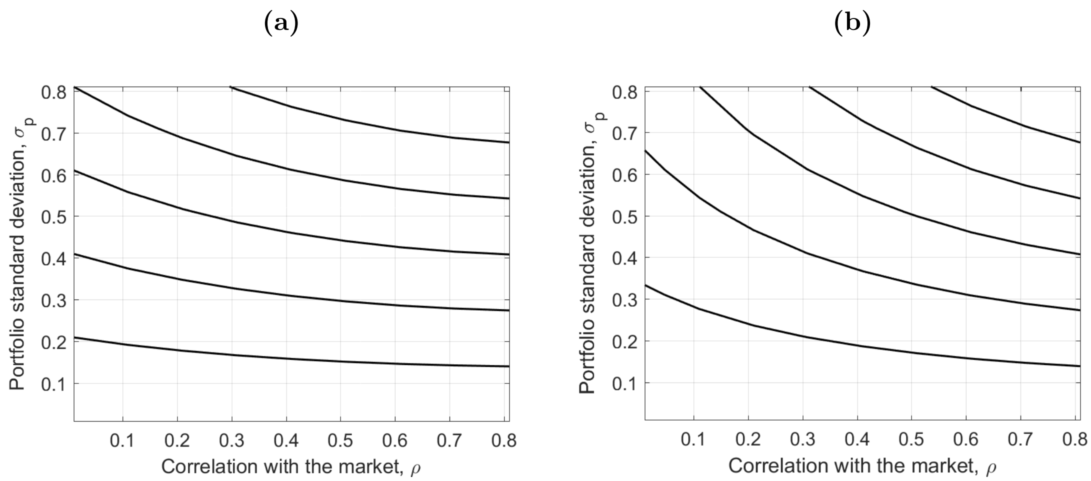

Figure 3: Contour lines of $CoER^{\leq}$ for $\sigma_m = 0.9$ and $\mu_p = \mu_m = 0$: (a) $q_m = 10\%$ and $q_p = 10\%$, (b) $q_m = 10\%$ and $q_p = 50\%$.

as the parameter q_p decreases, portfolio correlation with the system becomes less significant for the investor. This is because the most prominent role is then played by the portfolio standard deviation. Such a dependence pattern is similar for $CoER^{=}$, as already discussed after Proposition 2. Second, and most interestingly, Figure 3 highlights that the higher the standard deviation of portfolio returns, the larger the role played by the correlation of the portfolio with the system (as the standard deviation gets larger, the level curves become steeper). In this respect, the top-right corners of panels (a) and (b) are empirically relevant cases because correlation usually increases with volatility (see, for instance, Longin and Solnik (1995), Campa and Chang (1998), Roll (1988), and Black (1976)).

Investors are likely to invest in portfolios with low (and even negative) correlation with the

[6] We only illustrate the dependence of $CoER^{\leq}$ on q_p, because its dependence on q_m is the same as for $CoER^{=}$, and this has already been discussed.

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system if the system is in a downturn. This is precisely the behavior captured by $CoER^{\leq}$. By construction, $CoER^{\leq}$ considers portfolio returns that are smaller than $CoVaR_{q_p}$ in stressed market conditions. However, when the system is in a downturn, low correlation between the portfolio and the system implies a higher value of $CoVaR_{q_p}$ (since $\frac{\mathrm{d}CoVaR_{q_p}}{\mathrm{d}\rho} < 0$) which, in turn, implies higher expected returns in stressed markets. Therefore, conditioned on a systemic event $\{R_m \leq VaR_{q_m}\}$, low correlation with the system implies a higher portfolio's expected return.

Unlike $CoER^{=}$, there is no closed-form expression for the optimal portfolio weights w^* maximizing the criterion $CoER^{\leq}$ in (**P2**). Nonetheless, we can approximate it well using the following procedure. Since $\mu_p = w^\intercal\mu$, $\sigma_p^2 = w^\intercal\Sigma w$, and $\rho = \frac{w^\intercal\sigma}{\sigma_m\sqrt{w^\intercal\Sigma w}}$, we can develop a first-order Taylor expansion of the expression of $\overline{\lambda}(\rho; q_m, q_p)$ given in (2.9) to obtain the following approximation $CoER^{\leq}(\rho_0)$ for $CoER^{\leq}$:

$$
\begin{aligned}
CoER^{\leq}(\rho_0) &= \mu_p - \left(\overline{\lambda}(\rho_0; q_m, q_p) + \frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho}\Big|_{\rho_0}(\rho - \rho_0)\right)\sigma_p \\
&= \underbrace{w^\intercal\mu}_{\substack{\text{Portfolio}\\\text{Return}}} - \frac{1}{\sigma_m}\frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho}\Big|_{\rho_0} \underbrace{w^\intercal\sigma}_{\substack{\text{Portfolio-System}\\\text{Covariance}}} - \left(\overline{\lambda}(\rho_0; q_m, q_p) - \frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho}\Big|_{\rho_0}\rho_0\right)\underbrace{\sqrt{w^\intercal\Sigma w}}_{\substack{\text{Portfolio}\\\text{St.Dev.}}}
\end{aligned}
\tag{2.11}
$$

Based on the approximation (2.11), we can then develop an approximate formulation of the original problem (**P2**) as follows:

$$
\begin{aligned}
\max_{w} \quad & CoER^{\leq}(\rho_0) \\
\text{s.t.} \quad & w^\intercal\mathbf{1} = 1.
\end{aligned}
\tag{$\widetilde{\mathbf{P2}}$}
$$

We can easily recognize the similarity of (2.11) with (2.4), which implies that we can obtain a closed-form solution to ($\widetilde{\mathbf{P2}}$). The portfolio that maximizes the criterion ($\widetilde{\mathbf{P2}}$) is provided in the following proposition.

Proposition 4 *Under Assumption 2, the portfolio that maximizes $CoER^{\leq}(\rho_0)$ is given by*

$$
w^*(\rho_0) = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\intercal\Sigma^{-1}\mathbf{1}} - \frac{1}{\sqrt{\left(\tilde{\lambda}^2 - \tilde{\Delta}^\intercal\tilde{Q}^{-1}\tilde{\Delta}\right)(\mathbf{1}^\intercal\Sigma^{-1}\mathbf{1})}}\begin{bmatrix} (\tilde{Q}^{-1})^\intercal\tilde{\Delta} \\ -\mathbf{1}^\intercal\tilde{Q}^{-1}\tilde{\Delta} \end{bmatrix}
\tag{2.12}
$$

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provided that

$$\tilde{\lambda} > \sqrt{\tilde{\Delta}^{\mathsf{T}} \tilde{Q}^{-1} \tilde{\Delta}}, \tag{2.13}$$

where

$$\tilde{\lambda} = \overline{\lambda}(\rho_0; q_m, q_p) - \frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho}\Big|_{\rho_0} \rho_0$$

and the expressions of $\tilde{Q}$ *and* $\tilde{\Delta}$ *are given by* (A.17) *in the Appendix.*

The solution $w^*(\rho_0)$ of the approximation problem ($\widetilde{\mathbf{P2}}$) can be used to find the solution w^* of the original problem (**P2**), as shown in the next proposition.

Proposition 5 *Let* $\rho_0 = \rho(w^*(\rho_0))$*. Then the vector* w^* *is a solution to the problem* (**P2**) *if and only if it is a solution to the approximation problem* ($\widetilde{\mathbf{P2}}$)*.*

By the proposition above, the correlation of the optimal portfolio with the system is equivalent to the solution ρ_0 of the nonlinear fixed point equation $\rho(w^*(\rho_0)) = \rho_0$. In other words, the solution to the original problem (**P2**) has the same form as the solution to the approximate problem ($\widetilde{\mathbf{P2}}$) and is given by (2.12). The constraint (2.13) admits the same interpretation as in $CoER^{=}$ (see also Section 2).

The optimal portfolio $w^*(\rho_0)$ given by (2.12) consists of two components. The first component is the well-known global minimum variance portfolio (lowest risk portfolio, see Merton (1972)). The second component is the systemic risk adjustment to the minimum variance portfolio in the sense that it alters the minimum variance portfolio in such a way that the resulting portfolio yields the maximum of $CoER^{\leq}$.

3 Relation to Mean–Variance analysis

In this section we discuss how the problems (**P1**) and (**P2**) analyzed in Section 2 compare with the traditional mean–variance formulation. We say that a portfolio w belongs to the mean–variance

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boundary if, for some μ_p, it solves

$$\begin{aligned}
\min_{w} \quad & \sigma_p^2 = w^{\mathsf{T}} \Sigma w \\
\text{s.t.} \quad & w^{\mathsf{T}} \mathbf{1} = 1, \\
& w^{\mathsf{T}} \mu = \mu_p.
\end{aligned}$$

We start with the following result.

Proposition 6 *If all assets in the portfolio are uncorrelated with the system, then the portfolio that maximizes $CoER^{=}$ (or $CoER^{\leq}$) belongs to the mean–variance boundary.*

The result can be intuitively understood as follows. If returns are jointly normal, then the assumption of zero covariance is equivalent to independence. It then follows from (2.2) and (2.7) that $CoVaR_{q_p}$ is the same as VaR_{q_p}, and both $CoER^{=}$ and $CoER^{\leq}$ coincide with the Expected Shortfall measure proposed by Rockafellar and Uryasev (2002), that is, with $CVaR_{q_p} = \mathbb{E}[R_p | R_p \leq VaR_{q_p}]$. This is well-known to generate the mean–variance boundary when used as the objective function by an investor (see, for example, Alexander and Baptista (2004)).

In the next proposition we show that the problem of finding the maximum of $CoER^{=}$ (or $CoER^{\leq}$) is equivalent to the problem of finding the minimum variance portfolio, i.e., the weight vector minimizing $w^{\mathsf{T}} \Sigma w$, if additional constraints on the portfolio–system covariance and expected portfolio returns are imposed.

Proposition 7 *Under Assumption 2, the optimal portfolio that maximizes $CoER^{=}$ (or $CoER^{\leq}$) belongs to the efficient boundary (σ_p, μ_p, c_p) implied by the solution of the following optimization problem*

$$\min_{w} \quad \sigma_p^2 = w^{\mathsf{T}} \Sigma w \tag{3.1}$$

$$s.t. \quad w^{\mathsf{T}} \mathbf{1} = 1,$$

$$w^{\mathsf{T}} \mu = \mu_p, \tag{3.2}$$

$$w^{\mathsf{T}} \sigma = c_p. \tag{3.3}$$

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The efficient boundary (σ_p, μ_p, c_p) is specified by the equation

$$\sigma_p^2 = [1, \mu_p, c_p] \left(B\Sigma^{-1}B^\intercal\right)^{-1} [1, \mu_p, c_p]^\intercal$$

where $B = [\mathbf{1}, \mu, \sigma]^\intercal$ is of full rank. Furthermore, the following separation result holds: Any optimal portfolio subject to a given expected return and covariance with the system can be replicated by three portfolios that belong to the efficient boundary.

To intuitively understand Proposition 7, let us assume that we have obtained the portfolio that maximizes $CoER^{=}$, that is, the portfolio that solves the problem (**P1**).[7] Denote the expected return on this portfolio by μ_p^* and the covariance with the system by c_p^*. This implies that the constraints $w^\intercal\mu = \mu_p^*$ and $w^\intercal\sigma = c_p^*$ are trivially satisfied by the optimal portfolio and adding these constraints to the problem (**P1**) does not change the solution. However, with these additional constraints the problem (**P1**) is the same as the quadratic optimization problem (3.1) where we use μ_p^* and c_p^* on the right-hand side of (3.2) and (3.3), respectively. Then the expected returns and covariance with the system in the expression of $CoER^{=}$ given by (2.4) can be replaced by the constants μ_p^* and c_p^*, respectively. The problem of minimizing $CoER^{=}$ then boils down to that of finding the portfolio with minimum variance.

However, the portfolio that solves the quadratic optimization problem (3.1) does not necessarily solve problems (**P1**) and (**P2**). For example, the first component of the optimal portfolio (2.12) (minimum variance portfolio) never yields the maximum of $CoER^{\leq}$ because the second component of the optimal portfolio (2.12) involves a positive definite matrix Q^{-1} and, thus, is never zero. At the same time, it can be easily seen that the minimum variance portfolio can be a solution of the quadratic optimization problem (3.1). Therefore, the set of portfolios that solve problems (**P1**) and (**P2**) for various values of the quantile parameters q_m and q_p is a subset of the set of optimal porfolios that solve the quadratic optimization problem (3.1) for various values of μ_p and c_p.

The above arguments have the following financial implications. By solving problem (**P1**) (or (**P2**)), an investor selects a portfolio on the boundary (σ_p, μ_p, c_p) that performs well in a low return environment, that is, a portfolio that yields the maximum of $CoER^{=}$ (or $CoER^{\leq}$). In other words, not all portfolios on the boundary (σ_p, μ_p, c_p) are attractive from the systemic-risk perspective;

[7] The same argument also applies to the portfolio that maximizes $CoER^{\leq}$ (solves problem (**P2**)).

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to find such portfolios the investor solves (**P1**) (or (**P2**)). Since the boundary (σ_p, μ_p, c_p) given in Proposition 7 satisfies an additional constraint on the covariance between the portfolio and the system, as compared with the mean–variance boundary, one would expect the mean–variance boundary to lie above the boundary (σ_p, μ_p, c_p) in the (σ_p, μ_p)–space. Thus, in normal market conditions, the portfolios that are mean–variance efficient are expected to have higher expected returns for a given value of portfolio standard deviation σ_p.

We conclude this section with an example that illustrates that the portfolios that are optimal from the mean–variance perspective may be suboptimal when their correlation with the system is taken into account, or equivalently, when $CoER^{\leq}$ (or $CoER^{=}$) is used as a performance metric.

Example 2 *Let $q_m = q_p = 10\%$, $\sigma_m = 0.2$, $\mu_m = \mu_p = 0$, and consider the following two portfolios:*

- *Portfolio A with $\rho_A = 0.01$ and $\sigma_{p_A} = 0.7$ (low correlation/high variance portfolio)*
- *Portfolio B with $\rho_B = 0.4$ and $\sigma_{p_B} = 0.6$ (high correlation/low variance portfolio)*

We then have that $CoER_A^{\leq} = -1.24$ ($CoER^{=} = -1.23$) and $CoER_B^{\leq} = -1.40$ ($CoER^{=} = -1.27$) implying that, ceteris paribus, portfolio A is preferred even though it has a higher standard deviation.

4 Empirical Analysis

In this section, we test the performance of the portfolio models $CoER^{=}$ and $CoER^{\leq}$ analyzed in Section 2. We use stock price data from the US and Canadian financial markets. We choose the constituents of the S&P500 Financials Index for the US market, and of the S&P/TSX Capped Financial Index for the Canadian market, as of October 1, 2018. We restrict our analysis to firms for which a historical price sequence is available starting from January 4, 2000. This yields a total of 52 US and 19 Canadian companies. We define the systemic event as a severe market decline, captured by a significant drop of the broad market index (a proxy for the financial system). Our definition is similar to that of Brownlees and Engle (2017), who argue that the triggering systemic event of the financial crisis is the decline (40% drop over 6 months) of the broad market index. We proxy the market index with the MSCI World Index. We use daily Bloomberg data from January 4, 2000 until October 20, 2018, and the time series for the MSCI World Index are converted into the corresponding domestic currencies based on end-of-day exchange rates reported by Bloomberg.

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4.1 Estimation methodology

In this section we describe the estimation methodology. We estimate the expected returns of assets in the portfolio using the historical average of the returns over the entire sample, and the covariance matrix of returns using the GARCH–DCC model (see Engle (2002, 2009)). We explicitly indicate the time dependence of the variables by adding the subscript t; for example, we write $r_{i,t}$ to indicate the rate of return of stock i at time t.

We collect the logarithmic returns on the n stocks and the market index in the vector $\tilde{r}_t = (\tilde{r}_{1,t}, ..., \tilde{r}_{n+1,t})^\top$, where $\tilde{r}_{i,t} = \ln(1 + r_{i,t}),\ i = 1, ..., n + 1$, and the $(n + 1)$th return is the return on the market index. We assume that conditional on the information set $\mathcal{F}_{t-1}$ available at time $t - 1$, the returns have an (unspecified) distribution $\mathcal{D}$ with zero mean and time-varying covariance

$$\tilde{r}_t \mid \mathcal{F}_{t-1} \sim \mathcal{D}(0, D_t C_t D_t)$$

where D_t^2 is the diagonal matrix whose i-th entry on the main diagonal is the variance $\sigma_{i,t}^2$ of $\tilde{r}_{i,t}$. We use the GJR–GARCH model first proposed by Glosten et al. (1993) to model the dynamics of variances, that is,

$$\sigma_{i,t}^2 = \omega_{Vi} + \alpha_{Vi}\tilde{r}_{i,t-1}^2 + \gamma_{Vi}\tilde{r}_{i,t-1}^2 I_{i,t-1} + \beta_{Vi}\sigma_{i,t-1}^2,$$

where $I_{i,t} = 1$ if $\tilde{r}_{i,t} < 0$ and 0 otherwise. We prescribe the following dynamics for the correlation matrix of the volatility adjusted returns $\varepsilon_t = D_t^{-1}\tilde{r}_t$:

$$C_t = \operatorname{diag}(Q_t)^{-1/2} Q_t \operatorname{diag}(Q_t)^{-1/2},$$

where Q_t is the pseudo-correlation matrix. The Dynamic Conditional Correlation (DCC) model then specifies the following dynamics for the matrix Q_t:

$$Q_t = (1 - \alpha_C - \beta_C)\, C + \alpha_C \varepsilon_{t-1}\varepsilon_{t-1}^\top + \beta_C Q_{t-1},$$

where C is the unconditional correlation matrix. The model is typically estimated using a two-step quasi-maximum likelihood estimation procedure (see Engle (2009)). We will refer to the above

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model specification as GARCH–DCC. The GARCH–DCC methodology is widely used in financial time series analysis as this class of models is parsimonious and is able to capture well many stylized facts of financial data. Table 1 provides the summary statistics for GARCH–DCC model weekly parameter estimates.

Table 1: The quantiles of GARCH–DCC weekly parameter estimates for the firms and the market index based on the sample period January 6, 2006 through October 19, 2018. The values are reported in the form of (10%, 50%, 90%)–quantiles.

US

	Firms	Market Index
ω_V	(0.0144, 0.0329, 0.0905)	(0.0123, 0.0137, 0.0156)
α_V	(0.0072, 0.0288, 0.0625)	(0.0000, 0.0000, 0.0043)
β_V	(0.8767, 0.9209, 0.9536)	(0.9046, 0.9137, 0.9194)
γ_V	(0.0447, 0.0860, 0.1282)	(0.1303, 0.1376, 0.1464)
α_C	(0.0016, 0.0035, 0.0038)	(0.0016, 0.0035, 0.0038)
β_C	(0.9770, 0.9807, 0.9854)	(0.9770, 0.9807, 0.9854)

Canada

	Firms	Market Index
ω_V	(0.0113, 0.0255, 0.0932)	(0.0078, 0.0110, 0.0119)
α_V	(0.0110, 0.0508, 0.1141)	(0.0000, 0.0012, 0.0045)
β_V	(0.8403, 0.9245, 0.9538)	(0.9331, 0.9381, 0.9514)
γ_V	(0.0002, 0.0388, 0.0760)	(0.0779, 0.0895, 0.0950)
α_C	(0.0031, 0.0040, 0.0043)	(0.0031, 0.0040, 0.0043)
β_C	(0.9747, 0.9871, 0.9901)	(0.9747, 0.9871, 0.9901)

The summary statistics on parameter estimates given in Table 1 reveal that the point estimates of the GJR–GARCH parameters are in line with the typical GJR–GARCH parameter estimates for equity markets (see Engle (2002, 2009) and Brownlees and Engle (2017)). Notice also that the market index has a slightly higher value of the asymmetric coefficient γ_V that the stocks, which implies a higher sensitivity of the market to large increases in volatility when the value of the index drops.

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4.2 Sharpe ratios results

We measure the performance of the portfolio using the annualized Sharpe ratio of portfolio returns[8] conditioned on a drop of $C\%$ in the market index, that is,

$$\text{Sharpe ratio} = \frac{\hat{\mu}_p}{\hat{\sigma}_p}\Big|_{R_m<C},$$

where

$$\hat{\mu}_p = \frac{1}{T}\sum_{t=1}^{T}(w_t^*)^\intercal r_t, \quad \hat{\sigma}_p^2 = \frac{1}{T-1}\sum_{t=1}^{T}((w_t^*)^\intercal r_t - \hat{\mu}_p)^2,$$

and w_t^* is the vector of optimal portfolio weights computed based on the information (covariance matrix and expected returns) available in period t, r_t is the vector of stock returns in period t, and T is the number of periods when the portfolio is rebalanced. As we would like to measure the performance of the portfolio during periods of market downturns, we only consider portfolio returns associated with declines of the market index.

We first construct the portfolio in January 2006 and then rebalance it at a given frequency, weekly or monthly, until October 2018. Thus, in the case of weekly rebalancing the portfolio is constructed on January 6, 2006 and we have $T = 670$ rebalancing periods, while in the case of monthly rebalancing, we construct the portfolio on January 31, 2016 and have $T = 153$ rebalancing periods. We choose the following two specifications for the threshold C: (i) $C = 0$, i.e, rebalancing occurs when the market index experiences negative returns, and (ii) $C = -1.5\%(-6.7\%)$ for weekly (monthly) rebalancing, corresponding to a 40% decrease in the market index over a 6-month period. Although the specification (ii) is the one which better captures a systemic event (a significant drop in the market index), we also test our portfolios on less severe market declines which are represented by the specification (i).

We evaluate the Sharpe ratio out-of-sample. At time t, we construct the portfolio using only information available at time t (estimates of covariance matrix and expected returns), and evaluate the Sharpe ratio on the new data that become available at time $t+1$. At time $t+1$, we rebalance the

[8] A similar methodology to assess portfolio performance has been used by Ban et al. (2018) and DeMiguel et al. (2009), among others. Ban et al. (2018) applied machine learning to portfolio construction and DeMiguel et al. (2009) assessed the efficiency of $1/n$ portfolio strategy.

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portfolio to reflect the new information available at that time. For comparison purposes, we also evaluate the performance against two other performance criteria, namely the Markowitz Minimum Variance (MV) and equally-weighted ($1/n$) portfolios. These are widely used benchmarks which, like $CoER^{\leq}$ and $CoER^{=}$, admit closed form expressions and hence serve as a good comparison to assess the role played by systemic risk in portfolio selection. Moreover, the portfolio maximizing the $CoER^{\leq}$ criterion is related to the MV portfolio, because it is obtained by adjusting the latter to account for systemic risk (see Section 2.2, Equation (2.12)). The equally-weighted portfolio represents a well-diversified portfolio of assets, that has been shown to have a superior performance over several other well-known portfolios (see DeMiguel et al. (2009)).

We first consider the Sharpe ratios evaluated at times when the market index has negative returns, that is, when $R_m < 0$.[9] The annualized Sharpe ratios and standard deviations of the portfolios are shown in Table 2.

It follows from Table 2 that $CoER^{\leq}$ consistently outperforms the other portfolios in terms of Sharpe ratios. Notably, the standard deviation of the $CoER^{\leq}$ maximizing portfolios is not the smallest among the considered portfolios implying that higher Sharpe ratios are due to larger returns. The data in Table 2 confirm that the parameters q_m and q_p allow the investor to express his relative preferences between the covariance of the portfolio with the market and the standard deviation of the portfolio, as theoretically shown in Section 2. Smaller values of q_p imply a smaller standard deviation for the $CoER^{\leq}$ maximizing portfolio. A smaller value of q_m attributes higher importance to the covariance with the market in periods of market downturns, and thus the Sharpe ratio under both $CoER^{=}$ and $CoER^{\leq}$ is decreasing in q_m. Consistently with intuition, an increase in Sharpe ratio unambiguously results in increased portfolio standard deviation. These conclusions appear robust against the rebalancing frequency, as also shown in Table 3 where monthly rebalancing is considered. Most importantly, $CoER^{\leq}$ consistently outperforms the other portfolios.

[9] We remark that in all evaluated portfolios, the assumptions (2.6) and (2.13) on λ and $\overline{\lambda}$, respectively, were never violated.

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Table 2: Annualized Sharpe ratios and standard deviations (in brackets) of four weekly rebalanced portfolios spanning the period January 6, 2006 through October 19, 2018. We condition on the event that the market index has negative weekly returns ($R_m < 0$). The maximum values of the Sharpe ratios are highlighted in bold.

(a) US stocks

		$q_m = 0.5$	$q_m = 0.3$
$q_p = 0.2$	$CoER^{\leqq}$	**−1.5760** (0.1352)	**0.9201** (0.1865)
	$CoER^{=}$	−4.3394 (0.1419)	−3.5878 (0.1339)
	Min.Var.	−3.0941 (0.1329)	−3.0941 (0.1329)
	$1/n$	−4.4961 (0.2400)	−4.4961 (0.2400)
$q_p = 0.1$	$CoER^{\leqq}$	**−2.1437** (0.1324)	**−0.8743** (0.1420)
	$CoER^{=}$	−4.3397 (0.1419)	−3.7733 (0.1352)
	Min.Var.	−3.0941 (0.1329)	−3.0941 (0.1329)
	$1/n$	−4.4961 (0.2400)	−4.4961 (0.2400)

(b) Canadian stocks

		$q_m = 0.5$	$q_m = 0.3$
$q_p = 0.2$	$CoER^{\leqq}$	**−2.9468** (0.1651)	**−2.7941** (0.1686)
	$CoER^{=}$	−3.4438 (0.1566)	−3.2376 (0.1597)
	Min.Var.	−3.1399 (0.1613)	−3.1399 (0.1613)
	$1/n$	−3.8509 (0.1714)	−3.8509 (0.1714)
$q_p = 0.1$	$CoER^{\leqq}$	**−3.0180** (0.1637)	**−2.9143** (0.1658)
	$CoER^{=}$	−3.4437 (0.1566)	−3.2810 (0.1590)
	Min.Var.	−3.1399 (0.1613)	−3.1399 (0.1613)
	$1/n$	−3.8509 (0.1714)	−3.8509 (0.1714)

Next, we evaluate the Sharpe ratio under more extreme market scenarios, i.e., assuming that $R_m < -1.5\%$ for weekly and $R_m < -6.7\%$ for monthly rebalancing (see Table 4).

Notice that the Sharpe ratios in Table 4 are greater (in absolute terms) than the corresponding Sharpe ratios in Table 2. This is expected, because we are considering more extreme scenarios of market downturns. Noticeably, Table 4 indicates that $CoER^{\leqq}$ outperforms the other portfolios. The improvement of Sharpe ratios for $CoER^{\leqq}$ over the benchmark portfolios becomes significant as the quantile parameter q_m gets smaller. This is consistent with intuition, because small values of q_m imply that more extreme market returns are considered when constructing the optimal portfolio. Furthermore, as q_m decreases, the Sharpe ratio of $CoER^{\leqq}$ increases more relative to that of $CoER^{=}$. For example, fixing $q_p = 0.1$ and decreasing q_m from 0.5 to 0.3 in the case of US portfolios results in an increase of the Sharpe ratio for $CoER^{\leqq}$ of $-3.8081 + 1.7081 = -2.1000$. The corresponding

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Table 3: Annualized Sharpe ratios and standard deviations (in brackets) of four monthly rebalanced portfolios spanning the period January 6, 2006 through October 19, 2018. We condition on the market index having negative monthly returns ($R_m < 0$). The maximum values of the Sharpe ratios are highlighted in bold.

(a) US stocks

		$q_m = 0.5$	$q_m = 0.3$
$q_p = 0.2$	$CoER^{\leq}$	$-$**0.3017** (0.1620)	**0.6135** (0.2019)
	$CoER^{=}$	-1.7082 (0.1463)	-1.2607 (0.1460)
	Min.Var.	-0.9542 (0.1505)	-0.9542 (0.1505)
	$1/n$	-2.1366 (0.2005)	-2.1366 (0.2005)
$q_p = 0.1$	$CoER^{\leq}$	$-$**0.5316** (0.1567)	$-$**0.0498** (0.1694)
	$CoER^{=}$	-1.7083 (0.1463)	-1.3622 (0.1457)
	Min.Var.	-0.9542 (0.1505)	-0.9542 (0.1505)
	$1/n$	-2.1366 (0.2005)	-2.1366 (0.2005)

(b) Canadian stocks

		$q_m = 0.5$	$q_m = 0.3$
$q_p = 0.2$	$CoER^{\leq}$	$-$**1.2136** (0.1556)	$-$**1.1660** (0.1603)
	$CoER^{=}$	-1.3772 (0.1418)	-1.3120 (0.1469)
	Min.Var.	-1.2746 (0.1500)	-1.2746 (0.1500)
	$1/n$	-1.5158 (0.1267)	-1.5158 (0.1267)
$q_p = 0.1$	$CoER^{\leq}$	$-$**1.2362** (0.1535)	$-$**1.2034** (0.1566)
	$CoER^{=}$	-1.3772 (0.1418)	-1.3257 (0.1457)
	Min.Var.	-1.2746 (0.1500)	-1.2746 (0.1500)
	$1/n$	-1.5158 (0.1267)	-1.5158 (0.1267)

increase for $CoER^{=}$ is smaller and equal to $-7.7929 + 6.6887 = -1.1042$. The Sharpe ratios in the case of monthly rebalanced portfolios given in Table 5 are broadly consistent with the estimates in the case of weekly rebalancing. Nevertheless, Table 5 provides somewhat different results, e.g., $CoER^{\leq}$ does not outperform the other portfolios for Canadian markets. We explain this behavior by noticing that, for Canadian markets, a decline of the Canadian dollar-denominated return on the market index below the level -6.7% only occurred 5 times in the data sample. Correspondingly, the average Sharpe ratio results are rather insufficient to draw reliable conclusions as there are too few data points used in evaluating the average.

We next analyse the level of diversification of $CoER^{\leq}$ and $CoER^{=}$ optimal portfolios, and its dependence on the quantile parameters q_m and q_p. As in Goetzmann and Kumar (2008), we use the Sum of Squared Portfolio Weights (SSPW) to measure diversification. The SSPW at a certain

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Table 4: Annualized Sharpe ratios and standard deviations (in brackets) of four weekly rebalanced portfolios spanning the period January 6, 2006 through October 19, 2018. We condition on the event that the market index has weekly returns smaller than −1.5% ($R_m < -1.5\%$). The maximum values of the Sharpe ratios are in bold.

(a) US stocks

		$q_m = 0.5$	$q_m = 0.3$
$q_p = 0.2$	$CoER^{\leq}$	$\mathbf{-2.8761}$ (0.1700)	$\mathbf{1.0973}$ (0.2501)
	$CoER^{=}$	−7.7921 (0.1602)	−6.3399 (0.1569)
	Min.Var.	−5.4896 (0.1589)	−5.4896 (0.1589)
	$1/n$	−7.8203 (0.2772)	−7.8203 (0.2772)
$q_p = 0.1$	$CoER^{\leq}$	$\mathbf{-3.8081}$ (0.1639)	$\mathbf{-1.7081}$ (0.1826)
	$CoER^{=}$	−7.7929 (0.1602)	−6.6887 (0.1570)
	Min.Var.	−5.4896 (0.1589)	−5.4896 (0.1589)
	$1/n$	−7.8203 (0.2772)	−7.8203 (0.2772)

(b) Canadian stocks

		$q_m = 0.5$	$q_m = 0.3$
$q_p = 0.2$	$CoER^{\leq}$	$\mathbf{-4.7900}$ (0.2200)	$\mathbf{-4.5634}$ (0.2246)
	$CoER^{=}$	−5.5413 (0.2071)	−5.2268 (0.2120)
	Min.Var.	−5.0794 (0.2145)	−5.0794 (0.2145)
	$1/n$	−6.1703 (0.2185)	−6.1703 (0.2185)
$q_p = 0.1$	$CoER^{\leq}$	$\mathbf{-4.8958}$ (0.2178)	$\mathbf{-4.7415}$ (0.2209)
	$CoER^{=}$	−5.5413 (0.2071)	−5.2924 (0.2109)
	Min.Var.	−5.0794 (0.2145)	−5.0794 (0.2145)
	$1/n$	−6.1703 (0.2185)	−6.1703 (0.2185)

date t is given by

$$SSPW_t = \sum_{i=1}^{n} w_{it}^2, \tag{4.1}$$

where w_{it} is the weight of stock i in the portfolio in period t. A lower value of SSPW reflects a higher level of diversification. We show the results only for US portfolios with monthly rebalancing because the results for weekly rebalancing are very similar (the plots for the Canadian portfolios are given in the Appendix).

Figure 5 indicates that the diversification level of MV portfolios is always between that of $CoER^{\leq}$ and $CoER^{=}$ portfolios. A decrease in q_m makes the optimal portfolio less diversified (compare panels (a) and (b) of Figure 5, respectively with panels (c) and (d) of the same figure), whereas a decrease in q_p implies a more diversified optimal portfolio (compare panels (a) and (c) of Figure 5, respectively with panels (b) and (d) of the same figure). This can be understood as

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Table 5: Annualized Sharpe ratios and standard deviations (in brackets) of four monthly rebalanced portfolios spanning the period January 6, 2006 through October 19, 2018. We condition on the event that the market index has monthly returns smaller than −6.7% ($R_m < -6.7\%$). The maximum values of the Sharpe ratios are highlighted in bold.

(a) US stocks

		$q_m = 0.5$	$q_m = 0.3$
$q_p = 0.2$	$CoER^{\leq}$	**−1.5538** (0.2766)	**0.0659** (0.3538)
	$CoER^{=}$	−4.0990 (0.2104)	−3.2071 (0.2255)
	Min.Var.	−2.7377 (0.2406)	−2.7377 (0.2406)
	$1/n$	−6.1798 (0.2255)	−6.1798 (0.2255)
$q_p = 0.1$	$CoER^{\leq}$	**−1.9243** (0.2638)	**−1.0906** (0.2946)
	$CoER^{=}$	−4.1003 (0.2104)	−3.3969 (0.2219)
	Min.Var.	−2.7377 (0.2406)	−2.7377 (0.2406)
	$1/n$	−6.1798 (0.2255)	−6.1798 (0.2255)

(b) Canadian stocks

		$q_m = 0.5$	$q_m = 0.3$
$q_p = 0.2$	$CoER^{\leq}$	−3.8621 (0.2801)	−3.8650 (0.2890)
	$CoER^{=}$	−3.8872 (0.2465)	−3.8912 (0.2604)
	Min.Var.	**−3.8466** (0.2692)	**−3.8466** (0.2692)
	$1/n$	−7.0936 (0.1028)	−7.0936 (0.1028)
$q_p = 0.1$	$CoER^{\leq}$	−3.8596 (0.2759)	−3.8630 (0.2819)
	$CoER^{=}$	−3.8862 (0.2465)	−3.8895 (0.2576)
	Min.Var.	**−3.8466** (0.2692)	**−3.8466** (0.2692)
	$1/n$	−7.0936 (0.1028)	−7.0936 (0.1028)

follows. As discussed in Section 2, smaller q_p implies that the portfolio composition is driven more by the portfolio variance rather than correlation of the portfolio with the market. As a result, for small q_p the selection criterion becomes closer to the mean–variance objective. As q_m decreases, portfolio correlation with the market becomes the dominating component, and the portfolio selection criterion deviates from the mean–variance objective. As a result, the $CoER^{\leq}$ portfolio becomes significantly more concentrated than the MV portfolio as q_m decreases. Intuitively, the fact that $\frac{\mathrm{d}CoER^{\leq}}{\mathrm{d}\rho} < 0$ makes the optimal portfolio more concentrated on the stocks that have smaller correlation with the market. This further implies that during crises, an investor is willing to sacrifice diversification benefits to select few stocks that have low correlation with the market.

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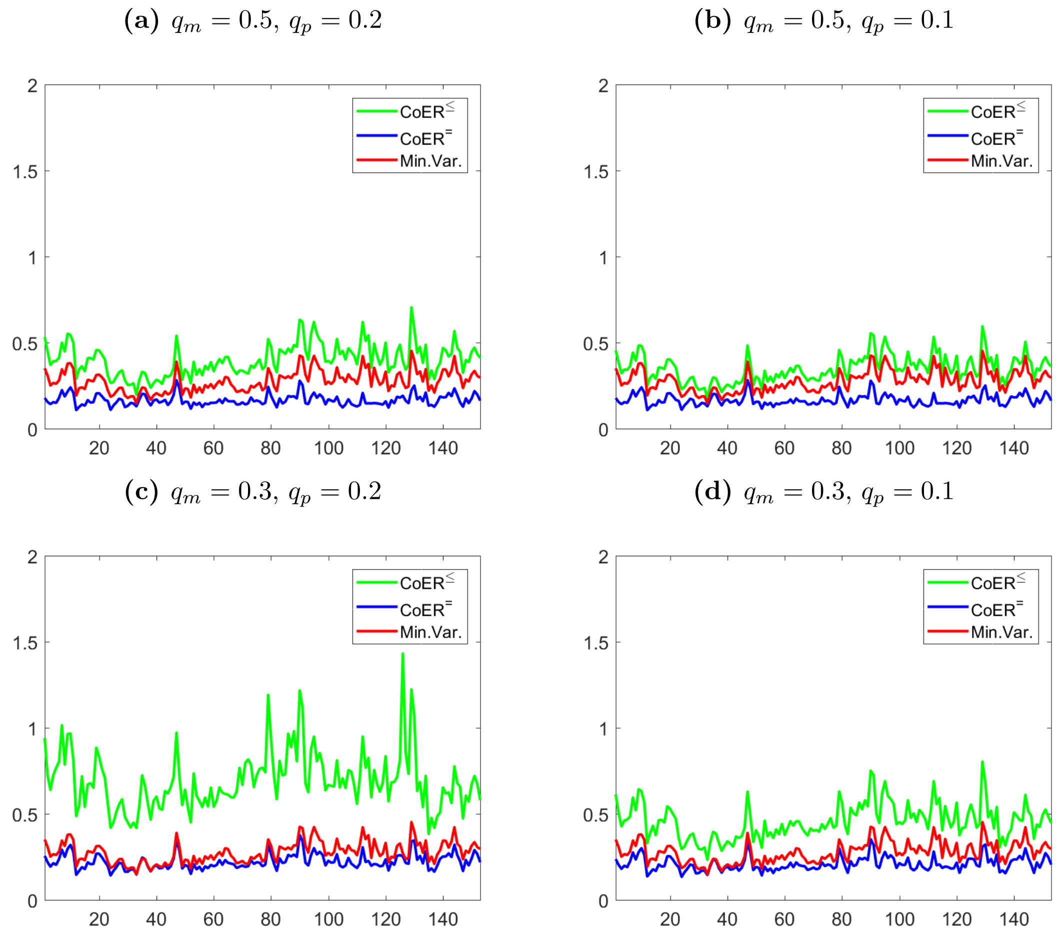

Figure 4: The Sum of Squared Portfolio Weights (SSPW) for portfolios based on data from the US market, for different values of the quantile parameters q_m and q_p. Rebalancing is executed monthly.

5 Conclusions

We have developed a model for portfolio selection in the presence of systemic risk. Our modeling approach defines a systemic event in terms of simultaneous bad performance of the market and of the portfolio's returns. We have used VaR and CoVaR measures to quantify the severity of the systemic event.

We have obtained an explicit solution to the portfolio selection problem, and studied the effect of the quantile parameters measuring tail events on the risk of the portfolio. We have shown that these quantile parameters allow the investor to properly balance portfolio variance and correlation with the system. Our analysis shows that any optimal portfolio with given expected return, variance, and covariance with the system can be replicated by three appropriately chosen portfolios. The

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optimal portfolio becomes mean–variance efficient if its constituents are not correlated with the system.

We have assessed the performance of our portfolio criterion on data from the Canadian and US equity markets. Our empirical analysis suggests that the optimal portfolio achieves higher Sharpe ratios than the Minimum Variance and $1/n$ portfolios during periods of market distress. Our empirical results also show that the optimal portfolio is less diversified than the Minimum Variance portfolio if it accounts for systemic risk. This is because, in anticipation of systemic events, the investor prefers to sacrifice diversification benefits and gain from the reduced exposure of the portfolio to elevated market distress.

A Proofs of Propositions

Proof of Proposition 1. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Using the definition of VaR

$$\int_{-\infty}^{VaR_q} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \mathrm{d}x = q,$$

we obtain that

$$VaR_q = \mu + \sigma\Phi^{-1}(q). \tag{A.1}$$

The Expected Shortfall is given by

$$\begin{aligned} ES &= \frac{1}{q}\int_{-\infty}^{VaR_q} \frac{x}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \mathrm{d}x \\ &= \mu - \frac{\sigma}{q}\phi\left(\Phi^{-1}(q)\right). \end{aligned} \tag{A.2}$$

Let $(X, Y) \sim \mathcal{BN}(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$. It is then well known (see for example, Section 4.7 of Bertsimas and Tsitsiklis (2000)) that

$$X|Y = y \sim \mathcal{N}\left(\mu_x + \rho\sigma_x\frac{y-\mu_y}{\sigma_y}, \sigma_x^2(1-\rho^2)\right) \tag{A.3}$$

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Let $R_p = X$ and $R_m = Y$. Since CoVaR is defined as the VaR of the portfolio when the system is at its VaR level (i.e., R_m is fixed at its VaR level $VaR_{q_m} = \mu_m + \sigma_m \Phi^{-1}(q_m)$), it follows from (A.1) and (A.3) that

$$CoVaR_{q_p} = \mu_p + \rho\sigma_p \Phi^{-1}(q_m) + \sigma_p \sqrt{1-\rho^2}\Phi^{-1}(q_p)$$

implying that $\frac{\mathrm{d}CoVaR_{q_p}}{\mathrm{d}\rho} < 0$ if and only if $\frac{\rho}{\sqrt{1-\rho^2}} < \frac{\Phi^{-1}(q_m)}{\Phi^{-1}(q_p)}$. Thus, we obtain (2.5).

It also follows from (A.2) and (A.3) that

$$\begin{aligned} CoER^{=} &= \underbrace{\mu_p + \rho\sigma_p \frac{VaR_{q_m} - \mu_m}{\sigma_m}}_{\mu \text{ in (A.2)}} - \underbrace{\frac{\sigma_p}{q_p}\sqrt{1-\rho^2}}_{\frac{\sigma}{q} \text{ in (A.2)}} \phi\left(\Phi^{-1}(q_p)\right) \\ &= \mu_p + \sigma_p \left(\rho\Phi^{-1}(q_m) - \frac{1}{q_p}\sqrt{1-\rho^2}\phi\left(\Phi^{-1}(q_p)\right)\right). \end{aligned} \tag{A.4}$$

The expession (2.4) follows from (A.4) by noticing that $\mu_p = w^\mathsf{T}\mu$, $\sigma_p^2 = w^\mathsf{T}\Sigma w$, and $\rho = w^\mathsf{T}\sigma/\sigma_m\sigma_p$.
□

Proof of Proposition 2. We can write the covariance matrix of returns on all stocks and the system as

$$\begin{bmatrix} \Sigma & \sigma \\ \sigma^\mathsf{T} & \sigma_m^2, \end{bmatrix},$$

where σ is the column vector of covariances of each stock with the system. The joint distribution of portfolio's return and the system return is bivariate normal with mean vector $[w^\mathsf{T}\mu, \mu_m]$ and covariance matrix

$$\begin{bmatrix} w^\mathsf{T}\Sigma w & w^\mathsf{T}\sigma \\ w^\mathsf{T}\sigma & \sigma_m^2 \end{bmatrix} = \begin{bmatrix} \sigma_p^2 & w^\mathsf{T}\sigma \\ w^\mathsf{T}\sigma & \sigma_m^2 \end{bmatrix}.$$

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Because the portfolio correlation with the system is $\rho = w^\intercal \sigma / \sigma_m \sigma_p$, it follows from (A.4) that

$$\begin{aligned} CoER^{=} &= \mu_p + \sigma_p \left(\rho \Phi^{-1}(q_m) - \frac{1}{q_p} \sqrt{1-\rho^2} \phi\left(\Phi^{-1}(q_p)\right) \right) \\ &= w^\intercal \mu + \sqrt{w^\intercal \Sigma w} \left(\frac{w^\intercal \sigma}{\sigma_m \sqrt{w^\intercal \Sigma w}} \Phi^{-1}(q_m) - \frac{1}{q_p} \sqrt{1 - \left(\frac{w^\intercal \sigma}{\sigma_m \sqrt{w^\intercal \Sigma w}} \right)^2} \phi\left(\Phi^{-1}(q_p)\right) \right) \\ &= w^\intercal \mu + \frac{\Phi^{-1}(q_m)}{\sigma_m} w^\intercal \sigma - \frac{\phi\left(\Phi^{-1}(q_p)\right)}{\sigma_m q_p} \sqrt{\sigma_m^2 w^\intercal \Sigma w - w^\intercal \sigma \sigma^\intercal w} \\ &= w^\intercal \left(\mu + \frac{\Phi^{-1}(q_m)}{\sigma_m} \sigma \right) - \frac{\phi\left(\Phi^{-1}(q_p)\right)}{\sigma_m q_p} \sqrt{\sigma_m^2 w^\intercal \left(\Sigma - \frac{1}{\sigma_m^2} \sigma \sigma^\intercal \right) w} \\ &= w^\intercal \widehat{\mu} - \lambda \sqrt{w^\intercal \widehat{\Sigma} w}, \end{aligned} \tag{A.5}$$

where

$$\widehat{\mu} = \mu + \frac{\Phi^{-1}(q_m)}{\sigma_m} \sigma, \quad \lambda = \frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}, \quad \widehat{\Sigma} = \Sigma - \frac{1}{\sigma_m^2} \sigma \sigma^\intercal. \tag{A.6}$$

From Landsman (2008,a) it follows that the maximum of (A.5) under the constraint $w^\intercal \mathbf{1} = 1$ is reached at

$$w^* = \frac{\widehat{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^\intercal \widehat{\Sigma}^{-1} \mathbf{1}} - \frac{1}{\sqrt{\left(\lambda^2 - \Delta^\intercal Q^{-1} \Delta\right) \left(\mathbf{1}^\intercal \widehat{\Sigma}^{-1} \mathbf{1}\right)}} \begin{bmatrix} (Q^{-1})^\intercal \Delta \\ -\mathbf{1}^\intercal Q^{-1} \Delta \end{bmatrix},$$

where

$$\Delta = \widehat{\mu}_n \mathbf{1} - \tilde{\widehat{\mu}}, \quad Q = \tilde{\Sigma} - \mathbf{1} \tilde{\sigma}^\intercal - \tilde{\sigma} \mathbf{1}^\intercal + \sigma_{nn} \mathbf{1} \mathbf{1}^\intercal \tag{A.7}$$

with $\tilde{\widehat{\mu}} = [\widehat{\mu}_1, ..., \widehat{\mu}_{n-1}]^\intercal$ and $\tilde{\Sigma}$, $\tilde{\sigma}$ defined by

$$\widehat{\Sigma} = \begin{bmatrix} \tilde{\Sigma} & \tilde{\sigma} \\ \tilde{\sigma}^\intercal & \sigma_{nn} \end{bmatrix}$$

provided that

$$\lambda > \sqrt{\Delta^\intercal Q^{-1} \Delta}. \tag{A.8}$$

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The restriction (A.8) guarantees that the solution is finite. □

Proof of $\frac{\mathrm{d}\lambda}{\mathrm{d}q_p} < 0$. We have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}q_p} = \frac{\mathrm{d}}{\mathrm{d}q_p}\left(\frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}\right) = -\frac{q_p\Phi^{-1}(q_p) + \phi\left(\Phi^{-1}(q_p)\right)}{q_p^2},$$

where we have used the inverse function theorem. Clearly, $\frac{\mathrm{d}}{\mathrm{d}q_p}\left(\frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}\right) < 0$ is equivalent to

$$q_p\Phi^{-1}(q_p) + \phi\left(\Phi^{-1}(q_p)\right) > 0. \tag{A.9}$$

Let $x = \Phi^{-1}(q_p)$. Then the inequality (A.9) becomes $f(x) := x\Phi(x) + \phi(x) > 0$. This holds true because $\frac{\mathrm{d}f}{\mathrm{d}x} = \Phi(x) > 0$ and $\lim_{x\to-\infty} f(x) = 0$, where we have applied L'Hospital's rule twice to the fraction $\frac{x}{1/\Phi(x)}$. □

Proof of Proposition 3. We use the results from Kan and Robotti (2017). If $(X_1, X_2) \sim \mathcal{BN}(m, \Sigma)$ where Σ is the correlation matrix, they provide an explicit expression for the first moment of the *lower* truncated random variable Z_1, that is,

$$\mathbb{E}[Z_1] := \mathbb{E}[X_1 | a_1 < X_1, a_2 < X_2] = m_1 + \frac{\phi(\eta_1)\Phi(w_{2\cdot1}) + \rho\phi(\eta_2)\Phi(w_{1\cdot2})}{\Phi_2(\eta_1, \eta_2; \rho)},$$

where

$$\begin{aligned} \eta_i &= m_i - a_i, \\ \Phi_2(\eta_1, \eta_2; \rho) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\eta_1}\int_{-\infty}^{\eta_2} \exp\left(-\frac{(x^2 + y^2 - 2\rho xy)}{2(1-\rho^2)}\right) \mathrm{d}x\,\mathrm{d}y, \\ w_{i\cdot j} &= (\eta_i - \rho\eta_j)/\sqrt{1-\rho^2}. \end{aligned} \tag{A.10}$$

The corresponding expression in the case of a general covariance matrix Σ can be obtained by replacing a_i with a_i/σ_i, m_i with m_i/σ_i, and multiplying everything by σ_1 (standard deviation of X_1).

To obtain the first moment of the *upper* truncated bivariate normal random variable, one should

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notice that $(-X_1, -X_2) \sim \mathcal{BN}(-m, \Sigma)$, and thus, replace m with $-m$, a with $-b$, and multiply the result by -1. Mathematically,

$$\mathbb{E}[X_1|X_1 < b_1, X_2 < b_2] = m_1 - \sigma_1 \frac{\phi(\eta_1)\Phi(w_{2\cdot 1}) + \rho\phi(\eta_2)\Phi(w_{1\cdot 2})}{\Phi_2(\eta_1, \eta_2; \rho)},$$

where $\eta_i = (b_i - m_i)/\sigma_i$. The result (2.8) follows from the fact that $\Phi_2(\eta_1, \eta_2; \rho) = q_m q_p$ in our setting.

Since the Gaussian cdf and pdf are positive functions and $\rho \geq 0$, we also have $\overline{\lambda}(\rho; q_m q_p) > 0$. Furthermore, from the definition of $CoVaR_{q_p}$, we have

$$\begin{aligned}
&\mathbb{P}\left(R_p \leq CoVaR_{q_p} \middle| R_m \leq VaR_{q_m}\right) = q_p \\
\Longleftrightarrow &\mathbb{P}\left(\frac{R_p - \mu_p}{\sigma_p} \leq \frac{CoVaR_{q_p} - \mu_p}{\sigma_p} \middle| \frac{R_m - \mu_m}{\sigma_m} \leq \frac{VaR_{q_m} - \mu_m}{\sigma_m}\right) = q_p \\
\Longleftrightarrow &\mathbb{P}\left(n_1 \leq \eta_1 \middle| n_2 \leq \eta_2\right) = q_p,
\end{aligned}$$

where the distribution of (n_1, n_2) is a bivariate standard Gaussian. Therefore, η_1 and η_2 are independent of μ_p, μ_m, σ_m, and σ_p implying that $\overline{\lambda}(\rho; q_m, q_p)$ is also independent of these parameters.

Next, we show that the function $\overline{\lambda}$ is strictly increasing in the correlation parameter ρ. Throughout this appendix, for notational compactness we will use X and Y in place of R_p and R_m, respectively. Thus, let $(X, Y) \sim \mathcal{BN}(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$, and define $\xi_1 := (x - \mu_x)/\sigma_x$, and $\xi_2 := (y - \mu_y)/\sigma_y$. It is well known that a bivariate normal pdf (see, for example, the Appendix in Sibuya (1960)) can be written as

$$\begin{aligned}
\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{(\xi_1^2 + \xi_2^2 - 2\rho\xi_1\xi_2)}{2(1-\rho^2)}\right) &= \frac{1}{\sigma_x\sigma_y\sqrt{1-\rho^2}}\phi\left(\frac{\xi_1 - \rho\xi_2}{\sqrt{1-\rho^2}}\right)\phi(\xi_2) && \text{(A.11)} \\
&= \frac{1}{\sigma_x\sigma_y\sqrt{1-\rho^2}}\phi\left(\frac{\xi_2 - \rho\xi_1}{\sqrt{1-\rho^2}}\right)\phi(\xi_1). && \text{(A.12)}
\end{aligned}$$

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Therefore, the bivariate Gaussian cdf may be rewritten as

$$\begin{aligned}
\mathbb{P}(X \leq a, Y \leq b) &= \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{\sigma_x \sigma_y \sqrt{1-\rho^2}} \phi\left(\frac{\xi_2 - \rho \xi_1}{\sqrt{1-\rho^2}}\right) \phi\left(\xi_1\right) \mathrm{d}y \, \mathrm{d}x \\
&= \int_{-\infty}^{\frac{a-\mu_x}{\sigma_x}} \int_{-\infty}^{\frac{b-\mu_y}{\sigma_y}} \frac{1}{\sqrt{1-\rho^2}} \phi\left(\frac{\xi_2 - \rho \xi_1}{\sqrt{1-\rho^2}}\right) \phi\left(\xi_1\right) \mathrm{d}\xi_2 \, \mathrm{d}\xi_1 \\
&= \int_{-\infty}^{\frac{a-\mu_x}{\sigma_x}} \frac{1}{\sqrt{1-\rho^2}} \phi\left(\xi_1\right) \int_{-\infty}^{\frac{b-\mu_y}{\sigma_y}} \phi\left(\frac{\xi_2 - \rho \xi_1}{\sqrt{1-\rho^2}}\right) \mathrm{d}\xi_2 \, \mathrm{d}\xi_1 \\
&= \int_{-\infty}^{\frac{a-\mu_x}{\sigma_x}} \phi\left(\xi_1\right) \int_{-\infty}^{\frac{\frac{b-\mu_y}{\sigma_y} - \rho\xi_1}{\sqrt{1-\rho^2}}} \phi\left(z\right) \mathrm{d}z \, \mathrm{d}\xi_1 \\
&= \int_{-\infty}^{\frac{a-\mu_x}{\sigma_x}} \Phi\left(\frac{\frac{b-\mu_y}{\sigma_y} - \rho\xi_1}{\sqrt{1-\rho^2}}\right) \phi\left(\xi_1\right) \mathrm{d}\xi_1. \qquad \text{(A.13)}
\end{aligned}$$

Let $a = CoVaR_{q_p}$ and $b = VaR_{q_m}$. Using (A.13), the definition of $CoVaR_{q_p}$ (see (2.7)), and the definition of VaR_{q_m} (see (2.1)), we obtain

$$\begin{aligned}
\mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{\sigma_x \sigma_y \sqrt{1-\rho^2}} \phi\left(\frac{\xi_2 - \rho \xi_1}{\sqrt{1-\rho^2}}\right) \phi\left(\xi_1\right) \mathrm{d}y \, \mathrm{d}x = q_m q_p \\
\iff \int_{-\infty}^{\frac{a-\mu_x}{\sigma_x}} \Phi\left(\frac{\frac{b-\mu_y}{\sigma_y} - \rho\xi_1}{\sqrt{1-\rho^2}}\right) \phi\left(\xi_1\right) \mathrm{d}\xi_1 = q_m q_p \\
\iff \int_{-\infty}^{\eta_1(\rho)} \Phi\left(\frac{\eta_2 - \rho\xi_1}{\sqrt{1-\rho^2}}\right) \phi\left(\xi_1\right) \mathrm{d}\xi_1 = q_m q_p, \qquad \text{(A.14)}
\end{aligned}$$

where we have used the definition of $\eta_1 = (a - \mu_x)/\sigma_x$ and $\eta_2 = (b - \mu_y)/\sigma_y$. Notice that η_2 is independent of ρ because marginal distributions of a bivariate normal distribution are independent of the correlation parameter, and it is only the marginal distribution of Y that is needed to evaluate VaR_{q_m}.

It is also well known (see, for example, the Appendix in Sibuya (1960)) that

$$\frac{\mathrm{d}}{\mathrm{d}\rho} \int_{-\infty}^{c_1} \Phi\left(\frac{c_2 - \rho z}{\sqrt{1-\rho^2}}\right) \phi\left(z\right) \mathrm{d}z = \phi_2(c_1, c_2; \rho), \qquad \text{(A.15)}$$

where c_1 and c_2 are given constants and ϕ_2 denotes the pdf of a standard bivariate normal distribution (see also (A.10) for the cdf of a standard bivariate normal distribution). Implicit differentiation

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of (A.14) with respect to ρ yields

$$\Phi\left(\frac{\eta_2-\rho\eta_1(\rho)}{\sqrt{1-\rho^2}}\right)\phi\left(\eta_1(\rho)\right)\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho}+\underbrace{\int_{-\infty}^{\eta_1(\rho)}\frac{\mathrm{d}}{\mathrm{d}\rho}\Phi\left(\frac{\eta_2-\rho\xi_1}{\sqrt{1-\rho^2}}\right)\phi\left(\xi_1\right)\mathrm{d}\xi_1}_{=\phi_2(\eta_1,\eta_2;\rho)}=0,$$

where we have applied (A.15) in the above expression. Thus, using (A.12) we obtain

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho}=-\frac{\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{\eta_2-\rho\eta_1}{\sqrt{1-\rho^2}}\right)}{\Phi\left(\frac{\eta_2-\rho\eta_1}{\sqrt{1-\rho^2}}\right)}. \tag{A.16}$$

In addition, it follows from the definition of $CoVaR_{q_p}$ and (A.16) that $\frac{\mathrm{d}CoVaR_{q_p}}{\mathrm{d}\rho}<0$.

Finally, we differentiate the expression of $\overline{\lambda}(\rho;q_m,q_p)$ given by (2.9) with respect to ρ. First, we evaluate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\eta_2-\rho\eta_1}{\sqrt{1-\rho^2}}\right)&=\frac{\rho\left(\eta_2-(1-\rho^2)\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho}\right)-\eta_1}{(1-\rho^2)^{3/2}},\\
\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\eta_1-\rho\eta_2}{\sqrt{1-\rho^2}}\right)&=-\frac{\eta_2-(1-\rho^2)\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho}-\rho\eta_1}{(1-\rho^2)^{3/2}},\\
\frac{\mathrm{d}}{\mathrm{d}x}\phi(x)&=-x\phi(x).
\end{aligned}$$

Differentiating the first term in the brackets of $\overline{\lambda}(\rho;q_m,q_p)$ (see its definition (2.9)), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\phi(\eta_1)\Phi\left(\frac{\eta_2-\rho\eta_1}{\sqrt{1-\rho^2}}\right)\right)=\underbrace{\frac{\rho}{\sqrt{1-\rho^2}}\phi(\eta_1)\phi\left(\frac{\eta_2-\rho\eta_1}{\sqrt{1-\rho^2}}\right)}_{=\rho\phi_2(\eta_1;\eta_2;\rho)}\left(\frac{\eta_2-\rho\eta_1}{1-\rho^2}-\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho}\right),$$

where we have used (A.16). As for the second term in the brackets of $\overline{\lambda}(\rho;q_m,q_p)$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi(\eta_2)\Phi\left(\frac{\eta_1-\rho\eta_2}{\sqrt{1-\rho^2}}\right)\right)=&\phi(\eta_2)\Phi\left(\frac{\eta_1-\rho\eta_2}{\sqrt{1-\rho^2}}\right)\\
&+\underbrace{\frac{\rho}{\sqrt{1-\rho^2}}\phi(\eta_2)\phi\left(\frac{\eta_1-\rho\eta_2}{\sqrt{1-\rho^2}}\right)}_{=\rho\phi_2(\eta_1;\eta_2;\rho)}\left(\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho}-\frac{\eta_2-\rho\eta_1}{1-\rho^2}\right).
\end{aligned}$$

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Hence, we deduce

$$\frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho} = \frac{1}{q_m q_p}\phi(\eta_2)\Phi\left(\frac{\eta_1 - \rho\eta_2}{\sqrt{1-\rho^2}}\right) > 0.$$

$\square$

Proof of Proposition 4. From Landsman (2008,a) it follows that the maximum of the objective function (2.11) under the constraint $w^{\mathsf{T}}\mathbf{1} = 1$ is reached at

$$w^* = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^{\mathsf{T}}\Sigma^{-1}\mathbf{1}} - \frac{1}{\sqrt{\left(\tilde{\lambda}^2 - \tilde{\Delta}^{\mathsf{T}}\tilde{Q}^{-1}\tilde{\Delta}\right)\left(\mathbf{1}^{\mathsf{T}}\Sigma^{-1}\mathbf{1}\right)}}\begin{bmatrix} (\tilde{Q}^{-1})^{\mathsf{T}}\tilde{\Delta} \\ -\tilde{\mathbf{1}}^{\mathsf{T}}\tilde{Q}^{-1}\tilde{\Delta} \end{bmatrix},$$

where

$$\tilde{\Delta} = \tilde{\mu}_n\tilde{\mathbf{1}} - \tilde{\tilde{\mu}}, \quad \tilde{\tilde{\mu}} = [\tilde{\mu}_1, ..., \tilde{\mu}_{n-1}]^{\mathsf{T}}, \quad \tilde{Q} = \overline{\Sigma} - \tilde{\mathbf{1}}\overline{\sigma}^{\mathsf{T}} - \overline{\sigma}\tilde{\mathbf{1}}^{\mathsf{T}} + \overline{\sigma}_{nn}\tilde{\mathbf{1}}\tilde{\mathbf{1}}^{\mathsf{T}}, \tag{A.17}$$

and

$$\Sigma = \begin{bmatrix} \overline{\Sigma} & \overline{\sigma} \\ \overline{\sigma}^{\mathsf{T}} & \overline{\sigma}_{nn} \end{bmatrix}$$

provided that

$$\tilde{\lambda} > \sqrt{\tilde{\Delta}^{\mathsf{T}}\tilde{Q}^{-1}\tilde{\Delta}}. \tag{A.18}$$

The restriction (A.18) ensures that the solution is finite. $\square$

Proof of Proposition 5. Assume that $\rho_0 = \rho(w^*(\rho_0))$, that is, ρ_0 is the correlation of the optimal portfolio with the system. The Lagrangian for the optimization problem (**P2**) is

$$L_1(w, \gamma) = CoER^{\leq} - \gamma(1 - w^{\mathsf{T}}\mathbf{1}),$$

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where γ is the Lagrange multiplier. Assume there exist w^* and γ^* such that $\nabla_w L_1(w^*, \gamma^*) = 0$, $\nabla_\gamma L_1(w^*, \gamma^*) = 0$, and $\rho_0 = \rho(w^*(\rho_0))$. Then the following conditions hold

$$\begin{aligned}
\nabla_w L_1(w^*, \gamma^*) &= \frac{\partial \mu_p}{\partial w}\Big|_{w^*} - \left(\frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho}\frac{\partial \rho}{\partial w}\sigma_p\right)\Big|_{w^*} - \overline{\lambda}(\rho_0; q_m, q_p)\frac{\partial \sigma_p}{\partial w}\Big|_{w^*} + \gamma^* \mathbf{1} = 0, \\
\nabla_\gamma L_1(w^*, \gamma^*) &= 1 - (w^*)^\mathsf{T}\,\mathbf{1} = 0.
\end{aligned}$$

Similarly, the Lagrangian for the approximation problem ($\widetilde{\mathbf{P2}}$) is given by

$$L_2(w, \gamma) = CoER^{\leq}(\rho_0) - \gamma(1 - w^\mathsf{T}\mathbf{1}).$$

It then follows that $\nabla_w L_1(w^*, \gamma^*) = 0$ and $\nabla_\gamma L_1(w^*, \gamma^*) = 0$ implies that $\nabla_w L_2(w^*, \gamma^*) = 0$ and $\nabla_\gamma L_2(w^*, \gamma^*) = 0$. This can be seen as follows:

$$\begin{aligned}
\nabla_w L_2(w^*, \gamma^*) &= \frac{\partial \mu_p}{\partial w}\Big|_{w^*} - \overline{\lambda}(\rho_0; q_m, q_p)\frac{\partial \sigma_p}{\partial w}\Big|_{w^*} - \left(\frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho}\frac{\partial \rho}{\partial w}\sigma_p\right)\Big|_{w^*} \\
&\quad - \frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho}\Big|_{\rho_0}\rho_0\frac{\partial \sigma_p}{\partial w}\Big|_{w^*} + \frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}\rho}\Big|_{\rho_0}\rho_0\frac{\partial \sigma_p}{\partial w}\Big|_{w^*} + \gamma^* \mathbf{1} = 0, \\
\nabla_\gamma L_2(w^*, \gamma^*) &= 1 - (w^*)^\mathsf{T}\,\mathbf{1} = 0.
\end{aligned}$$

Hence, the solution w^* for the exact formulation (**P2**) is also optimal for the approximate formulation ($\widetilde{\mathbf{P2}}$), and vice versa. □

Proof of Proposition 6. Write the covariance matrix of returns on all stocks and the system as

$$\begin{bmatrix} \Sigma & \sigma \\ \sigma^\mathsf{T} & \sigma_m^2, \end{bmatrix},$$

where σ is the column vector of covariances of each stock with the system. The joint distribution of

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portfolio's return and the system return is a bivariate Gaussian with mean $[w^\intercal \mu, \mu_m]$ and covariance

$$\begin{bmatrix} w^\intercal \Sigma w & w^\intercal \sigma \\ w^\intercal \sigma & \sigma_m^2 \end{bmatrix} = \begin{bmatrix} \sigma_p^2 & w^\intercal \sigma \\ w^\intercal \sigma & \sigma_m^2 \end{bmatrix}.$$

Since $\rho = w^\intercal \sigma / \sigma_m \sigma_p$, if the stocks are uncorrelated with the system then $\rho = 0$. Thus, for $CoER^{\leq}$ it follows that $\overline{\lambda}(\rho; q_m, q_p) = \overline{\lambda}(0; q_m, q_p)$, and it is independent of the portfolio weights w. Therefore, subject to the constraint $w^\intercal \mu = \mu_p$ we have

$$\max_w CoER^{\leq} = \min_w \overline{\lambda}(0; q_m, q_p)\sigma_p = \overline{\lambda}(0; q_m, q_p) \min_w \sigma_p.$$

For $CoER^{=}$, zero correlation implies $w^\intercal \sigma = 0$. Thus, subject to the constraint $w^\intercal \mu = \mu_p$, it follows from (2.4) that

$$\max_w CoER^{=} = \min_w \frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p} \sigma_p = \frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p} \min_w \sigma_p.$$

□

Proof of Proposition 7. We prove the result only for $CoER^{=}$ because the proof for $CoER^{\leq}$ is analogous due to the representation (2.11) and Proposition 5 (compare (2.11) with the $CoER^{=}$ given by (2.4)). Applying the result in Landsman (2008,b) we obtain that the constrained quadratic optimization problem (3.1) admits the closed-form solution

$$w^{**} = \Sigma^{-1} B^\intercal \left(B \Sigma^{-1} B^\intercal \right)^{-1} [1, \mu_p, c_p]^\intercal, \tag{A.19}$$

which yields the efficient boundary

$$\sigma_p^2 = (w^{**})^\intercal \Sigma w^{**} = [1, \mu_p, c_p] \left(B \Sigma^{-1} B^\intercal \right)^{-1} [1, \mu_p, c_p]^\intercal. \tag{A.20}$$

To show that the optimal portfolio w^* (see (2.12)) belongs to the efficient boundary, we substitute $(w^*)^\intercal \Sigma w^*$, $(w^*)^\intercal \mu$, $(w^*)^\intercal \sigma$, for σ_p^2, μ_p, c_p, respectively, in (A.20). It then follows directly that the equality in (A.20) holds.

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Next, we establish the separation result. Consider the portfolios specified by the vectors $p_i = [1, \mu_{p,i}, c_{p,i}]^\mathsf{T}$, $i = 1, 2, ..., N_p$ and assume we want to replicate a portfolio specified by $\bar{p} = [1, \bar{\mu}_p, \bar{c}_p]$. Let α_i be the proportion of wealth invested in each portfolio, then applying (A.19) we have

$$\sum_{i=1}^{N_p} \alpha_i w_i^{**} = \Sigma^{-1} B^\mathsf{T} \left(B \Sigma^{-1} B^\mathsf{T} \right)^{-1} \left(\sum_{i=1}^{N_p} \alpha_i p_i \right),$$

which is equal to

$$\Sigma^{-1} B^\mathsf{T} \left(B \Sigma^{-1} B^\mathsf{T} \right)^{-1} [1, \bar{\mu}_p, \bar{c}_p]^\mathsf{T}$$

if and only if

$$\sum_{i=1}^{N_p} \alpha_i p_i = \bar{p}. \tag{A.21}$$

It follows from the system (A.21) that the replication of portfolio $\bar{p}$ with these three portfolios ($N_p = 3$) is guaranteed provided that $\det([p_1, p_2, p_3]) \neq 0$. □

The Sum of Squared Portfolio Weights (SSPW) for Canadian portfolios. We present the results on the level of diversification in the $CoER^{\leq}$ and $CoER^{=}$ optimal portfolios, using data from the Canadian market (see Figure 5). □

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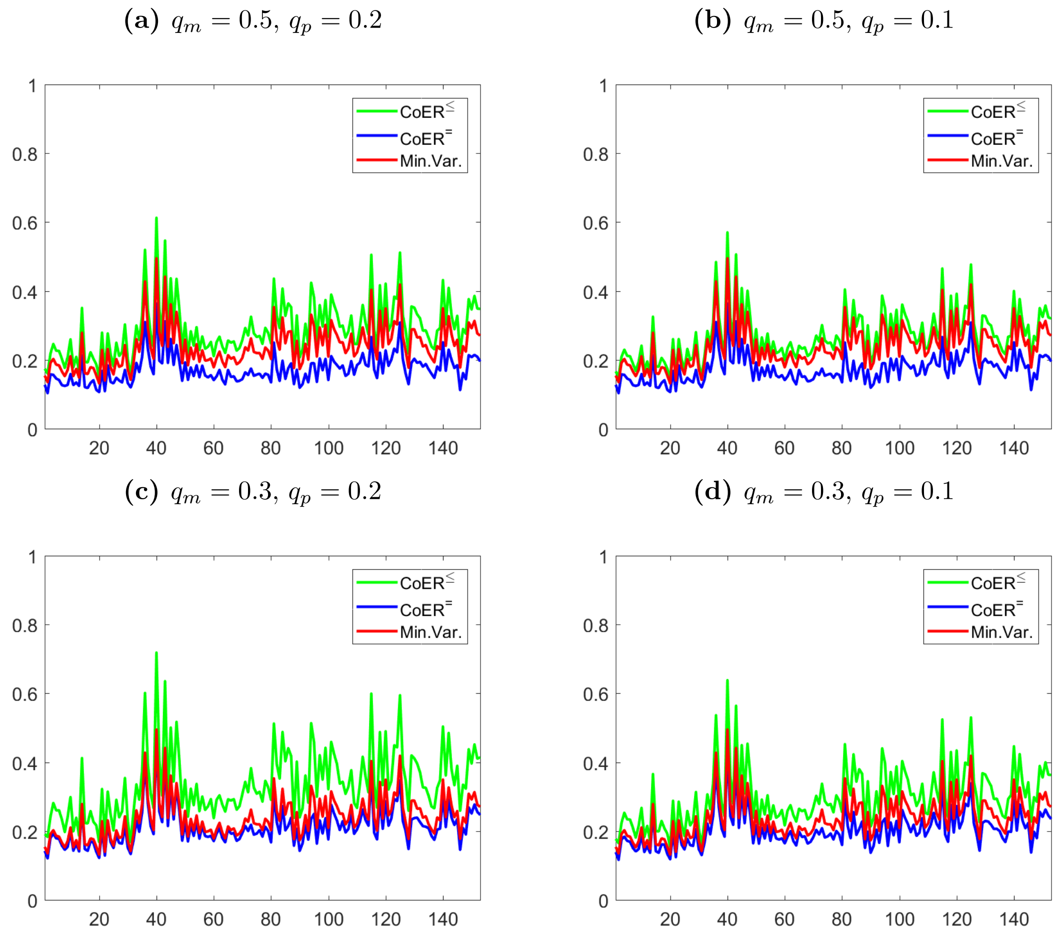

Figure 5: The Sum of Squared Portfolio Weights (SSPW) for portfolios based on data from the Canadian market, for different values of the quantile parameters q_m and q_p. Rebalancing is executed monthly.

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