

Dynamic Portfolio Decisions with Climate Risk and Model Uncertainty



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ABOUT THIS PAPER:

Failure to act on climate change ranked No. 1 on the list of top ten global risks by severity over the next ten years according to the World Economic Forum's annual Global Risks Report. It has become well understood that the impact of climate change on financial markets is highly uncertain and that our society is only learning about it over time. Notwithstanding existing research, little is known about the time horizon effects of climate uncertainty on optimal portfolio decisions, where climate uncertainty is defined as uncertainty about probabilities of future climate trajectories. This paper fills the gap and studies the impact of climate uncertainty on stock–bond–cash portfolios with short to medium term investment horizons.

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DYNAMIC PORTFOLIO DECISIONS WITH CLIMATE RISK AND MODEL UNCERTAINTY

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ABSTRACT

We study the effect of investment horizon on the optimal stock–bond–cash portfolio in a dynamic model with uncertainty about climate change. The stock risk premium is assumed to be an affine function of the average global temperature and an unobserved factor which is estimated via Bayesian learning. We assume that the probability distribution of future temperature is uncertain. The optimal investment strategy, robust to the uncertainty about climate change, is derived in closed form and analyzed for returns on the S&P500 index and the S&P500 ESG index. We find that stock market investment is quite sensitive to climate uncertainty with allocation to the S&P500 index being the most sensitive. We also show that, even for relatively short time horizons, welfare losses from climate uncertainty could be large for investments in either the S&P500 index or the S&P500 ESG index.

KEYWORDS

Portfolio choice; Climate change; Model uncertainty; Bayesian learning

1. Introduction

The last decade has witnessed a significant increase in research that investigates the effects of climate change on financial markets and, most notably, on financial stability (Engle et al. (2020), Bansal et al. (2017), Painter (2019), Hong et al. (2019), Carney (2015)). It has become well understood that the impact of climate change on financial markets is highly uncertain and that our society is only learning about it over time (Barnett et al. (2020), Daniel et al. (2019), Wagner and Zeckhauser (2017)). Notwithstanding existing research, little is known about the time horizon effects of climate uncertainty on optimal portfolio decisions. In this paper we study the impact of climate uncertainty on stock–bond–cash portfolios with short to medium term investment horizons.

Recent literature provides evidence that climate risks may impact companies through three channels: physical risks, climate policies and regulations, and change-related technological innovations (see Krueger et al. (2020), Kruttlı et al. (2019), Pankratz et al. (2019), Addoum et al. (2020)). To illustrate:

- (1) *physical risk*. For instance, insurance companies experience higher losses associated with property damage due to extreme weather events such as rising sea levels, droughts, etc.

- (2) *climate policies and regulations*. For example, environmental activists took Royal Dutch Shell to court demanding the energy firm drastically reduce the production of oil and gas to limit its effects on climate change. It is claimed that Royal Dutch Shell has broken Dutch law by knowingly hampering the global phase-out of fossil fuels, in a case that could force the company to reduce its CO₂ emissions.¹
- (3) *climate change-related technological innovations*. For instance, the emergence of electric vehicles could disrupt traditional car manufacturers. In turn, some companies benefit from climate risks (manufacturers of electronic vehicles for example). Although these considerations suggest that investors should integrate climate risk into their investment strategies, this can be quite challenging due to significant climate-related uncertainty (e.g., lack of well-established climate disclosure, ambiguity about appropriate climate hedge instruments). Therefore, when making investment decisions it is essential to consider both climate risk and climate uncertainty.

In our model we consider a market where investors can hold cash and trade in zero-coupon bonds and a stock index. The stock index risk premium is assumed to be an affine function of an observed and an unobserved variable. The observed variable is the global temperature change; the purpose of the unobserved variable is to capture the variation in the stock risk premium that is beyond the observed predictor. The extent to which temperature change can predict the risk premium is currently not well known. Hence investors must learn about the unobserved variable based on the observed stock prices, bond prices, and global temperature change.

Due to significant uncertainty about climate change (see Barnett et al. (2020), Daniel et al. (2019), Wagner and Zeckhauser (2017)), we also assume that investors are uncertain about the model for the temperature change dynamics. In particular, investors are uncertain about the probability distribution for global temperature change and, as a result, about the probability distribution of all other variables affected by climate change. Importantly, our flexible modelling framework allows us to study the impact of climate uncertainty on the stock market only as opposed to the economy more broadly. This is in line with recent empirical findings of Painter (2019) who reports, for example, that municipal bonds with maturities less than 25 years are not sensitive to climate risks. On the other hand, Bansal et al. (2017) and most recently Hong et al. (2019), find that climate change is an important factor that can influence stock returns.

We employ two approaches to study the impact of climate uncertainty: statistical and scenario based. In the statistical approach we apply the Method of Simulated Moments of Duffie and Singleton (1990) to estimate parameter values from the past data. In the scenario-based approach we adjust the long-run level of temperature change and its impact on the stock market to match future temperature projections typically given in climate science literature. We consider three possible scenarios for global average temperature increase in the next 100 years: 2°C (optimistic), 4°C (medium), and 6°C (pessimistic). Daniel et al. (2019) provide probability distribution of these scenarios under strict, modest, and ineffective mitigation policies. For example, the probability that the temperature change in the next 100 years will exceed 2°C is estimated to be 40% under a strict mitigation policy. However, this probability changes to 87% and 99% under a modest and an ineffective mitigation policies, respectively.

¹Shell Faces Climate Change Lawsuit in Dutch Court, Insurance Journal, December 1, 2020.

In this paper we make three contributions. First, we explore the impact of climate uncertainty on two types of stock indices: the S&P500 ESG index and the more traditional S&P500 index. The S&P500 ESG index is market-capitalization-weighted index designed to measure the performance of securities meeting sustainability criteria, while maintaining similar overall industry group weights as the S&P500 index. The primary motivation for analyzing Environmental, Social, and Governance (ESG) investment is the growing popularity of ESG considerations in portfolio construction (see Pedersen et al. (2020)). For instance, the 2018 Global Sustainable Investment Review reported that investments with explicit ESG goals exceeded \$30 trillion as of the beginning of 2018. However, as pointed out in Pedersen et al. (2020), there is still no clarity in regards to whether ESG considerations lower or increase expected portfolio returns. We study ESG investment from a different perspective: we explore the impact of climate uncertainty on ESG-oriented investment and compare it with more traditional investment in the S&P500 index. We find that investments in the S&P500 ESG index are less sensitive to climate uncertainty than investments in the S&P500 index. For example, when investors select the S&P500 ESG index, the optimal stock position could vary from 128% (low uncertainty) to 79% (high uncertainty) meaning a 38% decrease in the stock holdings, whereas for the S&P500 index the optimal stock position changes from 71% (low uncertainty) to 15% (high uncertainty) implying an almost 80% decrease in stock investments.

Second, we estimate the welfare loss (in terms of initial wealth) of an investor who ignores climate uncertainty when making investment decisions. In addition, we assess the decrease in welfare due to the mere presence of climate uncertainty. This latter loss analysis allows us to examine how detrimental it is to procrastinate implementing climate policies (i.e., maintain climate uncertainty). We show that both types of losses are quite large for the S&P500 and S&P500 ESG indices. For instance, even when uncertainty is low (high) for a 20-year investment horizon, the losses are about 10% (40%) for S&P500 and 80% (88%) for S&P500 ESG. The losses are similarly large in our scenario-based analysis. However, the loss from ignoring climate uncertainty is smaller for more adverse climate scenarios which is due to an already smaller investment in the stock and the bond in such scenarios.

Third, we study investment horizon effects on optimal portfolios and welfare losses under climate uncertainty. Although climate risk is usually discussed in the long-term context (see, for example, Giglio et al. (2018)), we show that welfare losses can be significant even for medium-term investors. For instance, when the investment horizon is only 10 years, ignoring climate uncertainty when making investment decisions could result in welfare loss as high as 53% for the S&P500 index and 63% for the S&P500 ESG index. Similarly, the loss from the mere presence of climate uncertainty is also substantial and is more than 29% for the S&P500 index and 39% for the S&P500 ESG index when uncertainty is high and the investment horizon is 10 years.

Finally, we provide a brief review of relevant literature on climate change and on portfolio choice under uncertainty. Engle et al. (2020) provide an econometric approach for constructing portfolios that hedge against climate change risk, but their methodology does not account for model uncertainty and learning about climate change. Andersson et al. (2016) develop a passive investment strategy based on low-carbon stock that allows long-term investors to hedge climate risk. Branger et al. (2013) and Escobar et al. (2016) consider a model where expected stock returns are predictable as observed and unobserved variables. However, both papers do not consider climate change risk. Furthermore, Escobar et al. (2016) ignores model uncertainty and Branger et al. (2013) assumes constant interest rate (that is, only stock-cash

portfolios). Kakeu (2017) provides a theoretical model of portfolio decisions that incorporates environmental externality concerns into investors' intertemporal wellbeing maximization goals. Fang et al. (2019) use the mean-variance framework to develop an approach for managing equity portfolios under climate change. Both, Kakeu (2017) and Fang et al. (2019) do not study the horizon effects on investment portfolios and do not explicitly model uncertainty about climate change.

This paper is further organized as follows. In Section 2 we formulate the model of optimal portfolio choice under climate uncertainty. The optimal investment strategy is derived in Section 3. Certainty equivalent costs are discussed in Section 4. In Section 5 we estimate model parameters and provide empirical analysis. Proofs are given in the Appendix.

2. Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{\mathcal{F}_t\}_{t \in [0, T]}$ a right-continuous filtration for this space such that $\mathcal{F}_T = \mathcal{F}$, where $0 < T < \infty$. All stochastic processes introduced below are defined on this probability space. We consider an investor who can trade in a stock (index), zero-coupon bonds, and a money market account (cash).

The dynamics of the money market account is given by:

$$dA_t = r_t A_t dt \quad (1)$$

where r_t is the stochastic short-term interest rate that is assumed to follow an Ornstein–Uhlenbeck process:

$$dr_t = \kappa_r(\bar{r} - r_t)dt - \sigma_r dW_t^P \quad (2)$$

where κ_r is the degree of mean reversion, \bar{r} is the long-run mean of the interest rate, $\sigma_r > 0$ is the interest rate volatility, and W_t^P is a Wiener process.

Stochastic interest rate suggests that a finite maturity bond is a non-redundant asset, and hence we can distinguish between cash and bonds. The interest rate process (2) implies (see Vasicek (1977)) that the price of a zero-coupon bond that matures at time T has the following dynamics:

$$dP_t = P_t \left((r_t + q)dt + \sigma_{P,t} dW_t^P \right) \quad (3)$$

where $q = q_r \sigma_{P,t}$ is the expected excess return on the bond, q_r is the constant market price of interest rate risk, and $\sigma_{P,t} = \sigma_r \frac{1}{\kappa_r} (1 - e^{-\kappa_r(T-t)})$ is the bond price volatility. For simplicity, we assume that the investor keeps the maturity of the bond in his portfolio constant (roll-over strategy), which implies that $\sigma_{P,t} \equiv \sigma_P$ (see also Chou et al. (2011) who make a similar assumption).

The evolution of the price S_t of the stock index is described by:

$$\frac{dS_t}{S_t} = \mu_{S,t} dt + \sigma_{S,t} dW_t^S \quad (4)$$

where $\mu_{S,t}$ is the expected stock return, $\sigma_{S,t}$ is the stock return volatility, and W_t^S is a Wiener process correlated with W_t^P ($dW_t^S dW_t^P = \rho_{SP} dt$).

Furthermore, in our modelling framework we assume that the stock risk premium, defined by $\frac{\mu_{S,t} - r_t}{\sigma_{S,t}}$, is an affine function of the *observed* temperature change λ_t and the *unobserved* factor β_t with the following dynamics:²

$$d\lambda_t = \kappa_\lambda(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda dW_t^\lambda, \quad (5)$$

$$d\beta_t = \kappa_\beta(\bar{\beta} - \beta_t)dt + \sigma_\beta dW_t^\beta, \quad (6)$$

where $\kappa_\lambda, \kappa_\beta$ are the degrees of mean reversion, $\bar{\lambda}, \bar{\beta}$ are the long-run means of the factors, $\sigma_\lambda, \sigma_\beta > 0$ are the factors volatilities, W_t^λ, W_t^β are Wiener processes and $dW_t^S dW_t^\lambda = \rho_{S\lambda} dt$, $dW_t^\lambda dW_t^P = \rho_{\lambda P} dt$, $dW_t^S dW_t^\beta = \rho_{S\beta} dt$, $dW_t^\lambda dW_t^\beta = \rho_{\lambda\beta} dt$, $dW_t^\beta dW_t^P = \rho_{P\beta} dt$. Our assumption of mean reversion of temperature change is similar to the assumption made in the model of Daniel et al. (2019) who assume that the temperature change process asymptotes to double the value of the defined temperature change in 100 years, thereby exhibiting mean reversion.

We emphasize that our modelling framework allows us to assume that the stock return volatility depends on the temperature change λ_t , that is, $\sigma_{S,t} = \sigma_S(\lambda_t)$ with function σ_S being uniformly positive.³ Our volatility assumptions are similar to that of Chacko and Viceira (2005) who model the stochastic volatility through specifying the process (factor) for the observed precision (inverse of volatility). In their model the stock risk premium is proportional to volatility. In contrast to their model we assume that the stock risk premium depends not only on the factor that influences volatility, but it also depends on an unobserved factor, that is, the stock returns follow:

$$\frac{dS_t}{S_t} = \left(r_t + \sigma_S(\lambda_t) \left(p + p_\lambda \lambda_t + p_\beta \beta_t \right) \right) dt + \sigma_S(\lambda_t) dW_t^S \quad (7)$$

where p, p_λ and p_β are constants with p_λ, p_β specifying the predictive power of λ_t and β_t , respectively, for the risk premium. Henceforth, we simplify the notation and write σ_S instead of $\sigma_S(\lambda_t)$.

Since the exact extent of temperature change impact on the stock market is currently far from being well known, it is natural to assume that investors learn about it over time. Our investor learns about the variations β_t in the stock risk premium beyond those that are caused by temperature change λ_t . In addition, unobservability of β_t highlights the fact that any predictor of expected stock returns is imperfect and will never be able to capture all variations in the stock risk premium. Thus, in Proposition 1 we derive the dynamics of all variables (as seen by the investor) based on the observed stock prices, temperature change, and interest rates.

Proposition 1. *Let $\mathcal{F}_t^{S,\lambda,r}$ be the smallest filtration with respect to which the observed processes S_t, λ_t , and r_t are measurable. The model for stock and bond prices, as seen by the investor, is:*

$$\begin{bmatrix} \frac{dS_t}{S_t} \\ \frac{dP_t}{P_t} \end{bmatrix} = \underbrace{\begin{bmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t + p_\beta \hat{\beta}_t) \\ r_t + q \end{bmatrix}}_{\mu} dt + \underbrace{\begin{bmatrix} -\frac{\sigma_S}{\sigma_P \rho_{SP}} & -\frac{0}{\sigma_P \sqrt{1 - \rho_{SP}^2}} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} dB_{t,1} \\ dB_{t,2} \end{bmatrix}}_{dB_t} \quad (8)$$

²Our assumption that the global temperature affects the stock risk premium is in line with a recent empirical study of Bansal et al. (2017).

³The assumption that the volatility does not depend on the unobserved factor β_t is necessary to apply the filtering techniques of Liptser and Shiryaev (2001) to obtain an estimate $\hat{\beta}_t$ of the unobserved predictor.

and the factors dynamics is:

$$\underbrace{\begin{bmatrix} d\lambda_t \\ dr_t \\ d\beta_t \end{bmatrix}}_{dY_t} = \underbrace{\begin{bmatrix} \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \\ \kappa_\beta(\bar{\beta} - \beta_t) \end{bmatrix}}_{\mu^Y} dt + \underbrace{\begin{bmatrix} \frac{\sigma_\lambda \rho_{S\lambda}}{A_1} & \frac{\sigma_\lambda \hat{\rho}_\lambda}{A_2} & \frac{0}{A_3} \\ -\sigma_r \rho_{SP} & -\sigma_r \hat{\rho}_{\lambda P} & -\sigma_r \hat{\rho}_P \end{bmatrix}}_{\Sigma^Y} \underbrace{\begin{bmatrix} dB_{1,t}^Y \\ dB_{2,t}^Y \\ dB_{3,t}^Y \end{bmatrix}}_{dB_t^Y} \quad (9)$$

where the processes B_t and B_t^Y are $\mathcal{F}_t^{S,\lambda,r}$ -adapted correlated Wiener processes with correlation matrix:

$$\rho = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\hat{\rho}_{\lambda P}}{\sqrt{1-\rho_{SP}^2}} \\ 0 & \frac{\hat{\rho}_P}{\sqrt{1-\rho_{SP}^2}} \end{bmatrix}, \quad (10)$$

where $\rho_{ij} = dB_{i,t}^Y dB_{j,t}^Y$ and where $A_1, A_2, A_3, \hat{\rho}_\lambda, \hat{\rho}_{\lambda P}, \hat{\rho}_P$ are given in the Appendix A

Proof. See Appendix A. \square

In order to derive the wealth dynamics of the investor we let π_t^S be the fraction of wealth invested in the stock, π_t^P be the fraction of wealth invested in the bond, and $(1 - \pi_t^S - \pi_t^P)$ be the fraction of wealth invested in the money market account. Let us denote $\phi_t := (\pi_t^S, \pi_t^P)^\top$, then the wealth X_t of the investor follows the dynamics:

$$dX_t = X_t \left(\phi_t^\top (\mu - r) + r_t \right) dt + X_t \phi_t^\top \Sigma dB_t \quad (11)$$

where $r = [r_t, r_t]^\top$.

The wealth dynamics (11) assumes complete certainty about the probability distribution of the underlying processes for the stock price, temperature change, and the interest rate. However, it has been widely recognized in academic literature on climate change that the future evolution of the global temperature is highly uncertain (see Barnett et al. (2020), Daniel et al. (2019), Wagner and Zeckhauser (2017), among others). This uncertainty stems primarily from regulatory uncertainty (see also Intergovernmental Panel on Climate Change (IPCC)'s Special Report (IPCC (2018))) and our rather limited knowledge of global temperature aspects of climate science.

As the next step in our modelling, we assume that our investor is uncertain about the probability distribution for models (8) and (9) which will be referred to as the *reference model*. The investor realizes that other models might represent reality better, i.e., the reference model can be misspecified in the sense of Anderson et al. (2003). When making an investment decision, the investor wants to consider this uncertainty and a set of alternative plausible models that we now specify. Thus, we let u_t be an \mathbb{R} -valued $\mathcal{F}_t^{S,\lambda,r}$ -progressively measurable process and define the Radon–Nikodým derivative process as:

$$\begin{aligned} \xi_t^u &= \mathbb{E} \left[\frac{d\mathbb{P}^u}{d\mathbb{P}} \middle| \mathcal{F}_t^{S,\lambda,r} \right] \\ &= \exp \left(- \int_0^t \left(\frac{(1 + k_1^2 + k_2^2)u_s^2}{2} ds - u_s (dB_{1,s}^Y + k_1 dB_{2,s}^Y + k_2 dB_{3,s}^Y) \right) \right) \end{aligned} \quad (12)$$

where k_1 and k_2 are constants. Let $k = [1, k_1, k_2]^\top$, then according to the Girsanov's theorem, the process:

$$\tilde{B}_t^Y = k \int_0^t u_s ds + B_t^Y$$

is a Wiener process with respect to the probability measure \mathbb{P}^u . Thus, under the measure \mathbb{P}^u the state dynamics of the model is given by:

$$dZ_t = \left(\mu - \Sigma \rho^\top k u_t \right) dt + \Sigma \rho^\top d\tilde{B}_t^Y, \quad (13)$$

$$dY_t = \left(\mu^Y - \Sigma^Y k u_t \right) dt + \Sigma^Y d\tilde{B}_t^Y, \quad (14)$$

where $dZ_t = [dS_t/S_t, dP_t/P_t]^\top$. We will refer to the models (13) and (14) as the *alternative models* indexed by the process u_t .⁴ The associated wealth process is:

$$dX_t = X_t \left(\phi_t^\top (\mu - r) + r_t \right) dt + X_t \phi_t^\top \Sigma \rho^\top \left(-k u_t dt + d\tilde{B}_t^Y \right). \quad (15)$$

We would like to discuss our uncertainty modelling assumptions. First, there is a single source of uncertainty in our economy – climate change uncertainty, which is represented by the process u_t . Second, given the global scope of climate change, it is quite natural to assume that climate uncertainty affects all variables in the economy, however the extent of this influence could vary from one variable to another. Therefore, we introduced two parameters: k_1 and k_2 . For example, if $k_1 = \frac{-\hat{\rho}_P A_1 + \rho_{SP} A_3}{\hat{\rho}_P A_2 - \hat{\rho}_{\lambda P} A_3}$ and $k_2 = \frac{-A_2 \rho_{SP} + \hat{\rho}_{\lambda P} A_1}{A_2 \hat{\rho}_P - \hat{\rho}_{\lambda P} A_3}$ (see Appendix A for definitions of the constants used in the fractions), then global temperature uncertainty affects only the stock market, and vice versa. Although this example is rather extreme, it allows us to focus on specific aspects of climate change uncertainty (e.g., impact on the stock market only).

Finally, the investor's objective is to find an investment strategy that maximizes his expected utility for the time interval $[0, T]$. We assume that the investor has the utility function that exhibits constant relative risk aversion. Let us denote $y := (\lambda, r, \hat{\beta})^\top$ as the vector of state variables. The reward function of the investor who chooses the investment strategy ϕ and the alternative model specified by process u_t is:

$$w^u(t, x, y, \phi) = \frac{1}{1 - \gamma} \mathbb{E}_{t, x, y}^{\mathbb{P}^u} [(X_T)^\gamma], \quad (16)$$

and the value function (indirect utility) as

$$v(t, x, y) = \sup_{\phi} \inf_u \left[w^u(t, x, y, \phi) + \mathbb{E}_t^{\mathbb{P}^u} \int_t^T \frac{u_s^2}{2\eta_s} ds \right] \quad (17)$$

where $\gamma > 1, \gamma \neq 0$ is the constant relative risk aversion and the term $\mathbb{E}_t^{\mathbb{P}^u} \int_t^T \frac{u_s^2}{2\eta_s} ds$ is a penalty term given by the (derivative of) relative entropy with η being a function of time and state variables. For analytical tractability, and to obtain wealth-independent

⁴Our approach to uncertainty modelling is also known in the literature as "ambiguity aversion modelling". See, for example, Xinfeng (2021).

portfolio weights, we assume (see also Maenhout (2004, 2006)) that:

$$\eta_t = \frac{\theta}{(1 - \gamma)v(t, x, y)} \quad (18)$$

where θ is an ambiguity aversion parameter. Large values of θ correspond to small values of the penalty term in (17) implying that the investor is more uncertain about the reference model.

3. Optimal Portfolio

We employ dynamic programming to solve the problem in Section 2. Following Merton (1971), the Hamilton–Jacobi–Bellman (HJB) equation, that the value function (17) should satisfy, is:

$$\begin{aligned} v_t + \sup_{\phi} \inf_u \left\{ \frac{1}{2} x^2 \phi^\top \Sigma \Sigma^\top \phi v_{xx} + x \left(\phi^\top (\mu - r) + r \right) v_x + x \phi^\top \Sigma \rho^\top \Sigma^{Y\top} v_{xy} \right. \\ \left. + \frac{1}{2} \text{Tr} \left(\Sigma^Y \Sigma^{Y\top} v_{yy} \right) + \mu^Y v_y - x \phi^\top \Sigma \rho^\top k u v_x - \left(\Sigma^Y k u \right)^\top v_y + \frac{u^2}{2\eta} \right\} = 0 \end{aligned} \quad (19)$$

where:

$$v_y = \begin{bmatrix} v_\lambda \\ v_r \\ v_{\hat{\beta}} \end{bmatrix}, \quad v_{xy} = \begin{bmatrix} v_{x\lambda} \\ v_{xr} \\ v_{x\hat{\beta}} \end{bmatrix}, \quad v_{yy} = \begin{bmatrix} v_{\lambda\lambda} & v_{\lambda r} & v_{\lambda\hat{\beta}} \\ v_{\lambda r} & v_{rr} & v_{r\hat{\beta}} \\ v_{\lambda\hat{\beta}} & v_{r\hat{\beta}} & v_{\hat{\beta}\hat{\beta}} \end{bmatrix}. \quad (20)$$

The following proposition provides the solution to the HJB equation as well as the optimal investment strategy and the worst-case perturbation process.

Proposition 2. *The solution to the HJB equation (19) is given by:*

$$v(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma} \exp \left(a(t) + b(t)^\top y + \frac{1}{2} y^\top Q(t) y \right) \quad (21)$$

where the function $a(t)$, the vector $b(t)$, and the matrix $Q(t)$ satisfy the ordinary differential equations derived in Appendix B. The optimal investment strategy is:

$$\begin{aligned} \phi_t^* = & \underbrace{\left(\Sigma \left[\gamma I + \theta \rho^\top k k^\top \rho \right] \Sigma^\top \right)^{-1} (\mu - r)}_{\text{myopic demand}} \\ & + \underbrace{\left(\Sigma^\top \right)^{-1} \left[I + \frac{\theta}{\gamma} \rho^\top k k^\top \rho \right]^{-1} \rho^\top \left[I - \frac{\theta}{1-\gamma} k k^\top \right] \Sigma^{Y\top} (b(t) + Q(t)y)}_{\text{intertemporal hedge demand}} \end{aligned} \quad (22)$$

and the worst-case perturbation process is:

$$u_t^* = \theta \left(k^\top \rho \Sigma^\top \phi_t^* + \frac{\gamma}{1-\gamma} (\Sigma^Y k)^\top (b(t) + Q(t)y) \right). \quad (23)$$

Proof. See Appendix B. □

As our investment opportunity set is stochastic, the optimal portfolio (22) consists of two components: myopic (speculative) and hedge (see also Merton (1971)). The hedge component vanishes as the investment horizon decreases to zero. As is well known, the hedge component protects the investor against adverse changes in global temperature (λ_t), unobserved predictor (β_t), and interest rate (r_t). We also notice that both components depend on the climate uncertainty parameter θ . In Section 5 we explore the quantitative effects of climate uncertainty on the optimal investment strategy.

4. Certainty Equivalent Costs

In this section we derive certainty equivalent costs to assess the effect of climate uncertainty on an investor's welfare. We estimate:

- (1) welfare loss of the investor who ignores uncertainty about future temperature change and invests as if there is no uncertainty, and
- (2) cost of climate uncertainty, that is, we find welfare loss of the investor due to presence of climate uncertainty.

The analysis of an investor's suboptimal behaviour (item (1) in the above list) will allow us to assess the importance of *explicit* climate uncertainty modelling when agents make investment decisions. Similarly, the estimate of climate uncertainty cost (item (2)) will allow us to determine the impact of climate uncertainty on an investor's welfare.

First, we discuss how we estimate the loss from suboptimal investment strategy. The equation satisfied by the value function $\tilde{v}(t, x, y)$ of the agent who invests according to a suboptimal investment strategy $\tilde{\phi}$ is given by:

$$\begin{aligned} \tilde{v}_t + \frac{1}{2} x^2 \tilde{\phi}^\top \Sigma \Sigma^\top \tilde{\phi} \tilde{v}_{xx} + x \left(\tilde{\phi}^\top (\mu - r) + r \right) \tilde{v}_x \\ + x \tilde{\phi}^\top \Sigma \rho^\top \Sigma^Y \tilde{v}_{xy} + \frac{1}{2} \text{Tr} \left(\Sigma^Y \Sigma^Y \tilde{v}_{yy} \right) + \mu^Y \tilde{v}_y \\ - \eta \tilde{\phi}^\top \Sigma \rho^\top k \left(\Sigma^Y k \right)^\top \tilde{v}_y x \tilde{v}_x - \frac{1}{2} \eta \left(\tilde{\phi}^\top \Sigma \rho^\top k x \tilde{v}_x \right)^2 - \frac{1}{2} \eta \left(\left(\Sigma^Y k \right)^\top \tilde{v}_y \right)^2 = 0. \end{aligned} \quad (24)$$

The investor who ignores temperature change uncertainty is equivalent to the investor who uses the investment strategy ϕ_t^* given in Proposition 2, but mistakenly assuming that there is no uncertainty, that is, $\theta = 0$. In Proposition 3 we solve the HJB equation (24) for a certain type of suboptimal investment strategies.

Proposition 3. *Assume that the suboptimal investment strategy has the form*

$$\tilde{\phi} = f(t) + F(t)y, \quad (25)$$

where $f(t)$ and $F(t)$ are continuous functions. The solution to the HJB equation (24) is given by:

$$\tilde{v}(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma} \exp \left(\tilde{a}(t) + \tilde{b}(t)^\top y + \frac{1}{2} y^\top \tilde{Q}(t) y \right) \quad (26)$$

where the function $\tilde{a}(t)$, the vector $\tilde{b}(t)$, and the matrix $\tilde{Q}(t)$ satisfy the ordinary differential equations derived in Appendix B.

Proof. See Appendix B. □

We measure the loss incurred by the investor who follows a suboptimal investment strategy in terms of the initial wealth. Since $\tilde{v} \leq v$ and $v_x > 0$, we solve the following equation:

$$\tilde{v}(0, x, y) = v(0, x(1 - L^{sub}), y) \quad (27)$$

for $L^{sub} > 0$. Thus, the value of L^{sub} is equivalent to the percentage of the initial wealth that an optimally-behaved investor has to sacrifice in order to achieve the same (low) utility as the investor who invests in a suboptimal way. From (21) and (26) it follows that

$$L^{sub} = 1 - \exp \left(\frac{\gamma}{1 - \gamma} \left[\tilde{a}(t) - a(t) + (\tilde{b}(t) - b(t))^T y + \frac{1}{2} y^T (\tilde{Q}(t) - Q(t)) y \right] \right). \quad (28)$$

We evaluate the values of L^{sub} for the suboptimal investment strategy when investors ignore climate change uncertainty.

Now we describe how we estimate the impact of climate uncertainty on investor's welfare. To assess the impact of climate uncertainty on investor's welfare, we compare the value functions (maximum expected utilities) achieved by investors *with* and *without* climate change uncertainty. In other words, we solve for L^{unc} the equation

$$v^{\theta=\hat{\theta}}(0, x, y) = v^{\theta=0}(0, x(1 - L^{unc}), y) \quad (29)$$

where $v^{\theta=0}$ ($v^{\theta=\hat{\theta}}$) is the expected utility achieved by the investor who invests optimally when there is no (there is) climate change uncertainty. Since $v^{\theta=0} \geq v^{\theta=\hat{\theta}}$, we also have $L^{unc} \geq 0$ and from (21) we obtain

$$L^{unc} = 1 - \exp \left(\frac{\gamma}{1 - \gamma} \left[a^{\theta=\hat{\theta}}(t) - a^{\theta=0}(t) + (b^{\theta=\hat{\theta}}(t) - b^{\theta=0}(t))^T y + \frac{1}{2} y^T (Q^{\theta=\hat{\theta}}(t) - Q^{\theta=0}(t)) y \right] \right) \quad (30)$$

where the superscripts of functions a , b , and Q indicate the assumption made about the value of θ used to evaluate the functions (see Proposition 2). The value of L^{unc} will allow us to evaluate the impact of climate change uncertainty on an investor's welfare.

5. Empirical Analysis

We estimate the model from Section 2 by applying the Method of Simulated Moments of Duffie and Singleton (1990) to two sets of monthly time series from May 2005 to April 2021:

- (1) S&P500 index;⁵

⁵The S&P500 data are from Robert Shiller's website <http://www.econ.yale.edu/shiller/data.html>

- (2) S&P500 ESG index which is market-capitalization-weighted index designed to measure the performance of securities meeting sustainability criteria, while maintaining similar overall industry group weights as the S&P500 index.⁶

We consider two different indices to examine whether there are any welfare benefits regarding investment into the S&P500 ESG index as opposed to the more traditional S&P500. As mentioned in our Introduction, the main reason for analysing ESG investment is the growing popularity of ESG considerations in portfolio construction (see Pedersen et al. (2020)). In Appendix D we elaborate on the estimation procedure.

5.1. Data and estimated parameters

We obtain the 3-month T-bills data from the FRED website.⁷ Temperature data (in degrees Celsius) is obtained from the Berkeley Earth website and represent land-surface average temperature anomalies relative to the 1951–1980 average (i.e., temperature above the 1951–1980 average).⁸

Annualized parameter estimates for the S&P500 index and the state variables, are given in Table 1.

See Table 1.

From Table 1 it follows that the annual expected return and volatility of S&P500 index are 7.98% and 14.91%, respectively (assuming $r_t = \bar{r}$, $\lambda_t = \bar{\lambda}$ and $\beta = \bar{\beta}$). The long-run mean $\bar{\lambda}$ of the temperature change process λ_t is $0.0012C^\circ$ which implies that by the end of this century the temperature is expected to rise by $0.1C^\circ$ relative to 2020 (0.0012×80). We notice that this estimate is quite optimistic relative to temperature projections usually considered in the literature (see Daniel et al. (2019), IPCC (2014)). In this respect we would like to note that this estimate is consistent with the temperature data that we used for estimation in the sense that the average temperature change (per year) for our data is $0.0009C^\circ$. Furthermore, our uncertainty modelling approach allows us to consider more extreme scenarios of temperature change (see the discussion in Section 5.3). The long-run level \bar{r} of 3-month interest rate is 0.1% and we also observe a negative relationship ($\rho_{\lambda P} = -0.0651$) between bond price dynamics and temperature change which agrees with a recent study of Huynh and Xia (2021). Although rather small (in absolute terms), the negative correlation between stock and temperature change ($\rho_{S\lambda} = -0.0124$) indicates that the risk of global warming has an adverse impact on the stock market. This negative relationship also agrees with Bansal et al. (2017) who show that equity portfolios have negative exposures to long-run temperature fluctuations.

Annualized parameter estimates for the S&P500 ESG index and the state variables are provided in Table 2.

See Table 2.

The estimated annual expected return and volatility of S&P500 ESG index are 13.64% and 14.58%, respectively (assuming $r_t = \bar{r}$, $\lambda_t = \bar{\lambda}$ and $\beta = \bar{\beta}$). The long-run temperature change is $0.0012C^\circ$ which is the same as for the S&P500 index. Negative

⁶The data for the S&P500 ESG index are from Bloomberg. For the description of the index see <https://www.spglobal.com/spdji/en/indices/esg/sp-500-esg-index/#overview>

⁷See <https://fred.stlouisfed.org/>

⁸<http://berkeleyearth.org/data/>

correlation of the temperature change dynamics and returns on the S&P500 ESG index implies an adverse effect of global warming on ESG investments represented by the S&P500 ESG index.

5.2. Detection-Error Probabilities

To determine specific plausible values of the climate uncertainty parameter θ , we employ Detection-Error Probabilities (DEPs), $\varepsilon_N(\theta)$, that are used to measure the similarity between the reference and alternative models. Building on Chernoff (1952) and Newman and Stuck (1979), Anderson, Hansen, and Sargent (2003) advocate using θ such that the DEP is not less than 0.1. The value of θ that yields $\varepsilon_N(\theta) > 0.1$ makes it difficult for the investor to distinguish the reference and the worst-case models statistically; otherwise, the correct model could be easy to determine.

The worst-case model considered by the robust investor is given by (13)–(14) with $u_t = u_t^*$, where u_t^* is defined in (23). The Radon–Nikodým derivative between the reference model and the optimal worst-case model is $\Xi_{u,t} \triangleq E^{\mathbb{P}} \left[\frac{d\mathbb{P}^{u^*}}{d\mathbb{P}} | \mathcal{F}_t^{S,Z,r} \right]$. If model \mathbb{P} is the true model, the investor will reject it mistakenly for model \mathbb{P}^{u^*} based on a sample of size N when $\ln \Xi_{u,N} > 0$. The probability of this occurring is the first component in the definition of the detection-error probability. On the other hand, if model \mathbb{P}^{u^*} is the true model, the investor will reject it mistakenly for model \mathbb{P} based on a sample of size N when $\ln \Xi_{u,N} < 0$. The probability of this occurring is the second component in the definition of the detection-error probability. Thus, we define the DEP as

$$\varepsilon_N(\theta) = \frac{1}{2} \Pr(\ln \Xi_{u,N} > 0 \mid \mathbb{P}, \mathcal{F}_0) + \frac{1}{2} \Pr(\ln \Xi_{u,N} < 0 \mid \mathbb{P}^{u^*}, \mathcal{F}_0),$$

where

$$\Xi_{u,t} = \exp \left(- \int_0^t (u_s^* dB_{1,s}^Y + k_1 u_s^* dB_{2,s}^Y + k_2 u_s^* dB_{3,s}^Y) - \frac{1 + k_1^2 + k_2^2}{2} \int_0^t (u_s^*)^2 ds \right),$$

To evaluate the DEP, we use Fourier inversion of the characteristic function of the Radon–Nikodým derivative (see Appendix E).

Since our analysis is based on two time series data, we evaluate two sets of DEPs for $N = 16$ years of observation. We also evaluate DEPs for $N = 25$ and $N = 40$ that would correspond to the time series available since the Kyoto Protocol and the time since climate change came into broader awareness, respectively. Calculated DEPs are shown in Table 3.

See Table 3.

As should be expected, DEPs are decreasing in climate uncertainty parameter θ . It also follows from Table 3 that plausible values of θ are smaller for the S&P500 ESG index. As mentioned before, the values of θ that yield DEPs more than 0.1 make it difficult for the investor to tell the reference and the alternative models apart. It can also be observed that DEPs are decreasing in N .⁹ This implies that the results for

⁹This pattern does not hold for $\varepsilon_{25}(1) = 0.3597$ and $\varepsilon_{16}(1) = 0.3563$ for S&P500 index. However, this outcome is due to error of numerical integration used to evaluate DEPs.

$N = 25$ and $N = 40$ can be inferred from the results for $N = 16$ that we consider as the baseline in our analysis. Indeed, the values of θ with DEPs exceeding 0.1 for $N = 25$ and $N = 40$ also have DEPs greater than 0.1 when $N = 16$.

We note that many state variables in our model make it rather challenging to accurately estimate DEPs that require numerical integration over the interval $[0, \infty)$. To ensure calculation accuracy, we verified the results using two integration techniques: Gauss–Laguerre quadrature (with 180 weights) and the trapezoid rule.

5.3. Worst-case returns and temperature changes

As follows from Equations (13)–(14), model uncertainty is equivalent to specific adjustments added to expected stock return and long-run temperature change. Figure 1 shows the worst-case adjustments for expected stock returns and long-run temperature change as a function of investment horizon ($T - t$) and climate uncertainty (θ).

See Figure 1.

The uncertainty adjustments for the long-run temperature change and for expected stock returns are of similar magnitude for both indices. In addition, the adjustments increase (in absolute terms) when uncertainty increases (large values of θ). When climate uncertainty is high ($\theta = 20$ for S&P500 and $\theta = 4$ for S&P500 ESG), the expected stock returns on both indices decrease by about 7%. The adjustments for the long-run temperature change alleviate the issue with a rather small estimate for long-run temperature change ($\bar{\lambda} = 0.0012$) discussed in Section 5.1. In particular, when $\theta = 20$ in the S&P500 index case (high level of climate model uncertainty) the optimal adjustment to $\bar{\lambda}$ induced by climate uncertainty is 0.0068 implying that the long-run end-of-century temperature change is $(0.0012 + 0.0068) \times 80 = 0.64C^\circ$ compared with $0.1C^\circ$ under the complete certainty case (i.e., $\theta = 0$). Thus, our uncertainty modelling approach allows us to consider long-run temperature changes that are higher than the estimated baseline long-run temperature change.

Importantly, a decrease in the long-run temperature change is favorable for the S&P500 and S&P500 ESG investment portfolios. This observation follows from Figure 1(A) because the presence of climate uncertainty implies an increase in the long-run temperature change: all long-run temperature adjustments are positive (black lines in Figure 1(A) and (B)). This result can be explained as follows. Since the bond is negatively correlated with temperature change ($\rho_{\lambda P} = -0.0651$) and the stock is negatively correlated with temperature change ($\rho_{S\lambda} = -0.0124$), innovations to the temperature that yield temperature increases usually decrease stock and bond returns. As a result, for the estimated parameter values (see Table 1), the portfolio gains in value from bond and stock positions when the temperature decreases.

5.4. Optimal portfolios

Although there are many values of θ that can be used to model climate uncertainty, in our analysis we will use three specifications that will correspond to low, medium, and high uncertainty about climate change. We assume:

- **S&P500 ESG index:** $\theta = 1$ (low), $\theta = 3$ (medium), $\theta = 4$ (high)
- **S&P500 index:** $\theta = 1$ (low), $\theta = 10$ (medium), $\theta = 20$ (high)

The specified values of θ for low, medium, and high levels of model uncertainty for the

S&P500 and S&P500 ESG indices roughly correspond to similar values for DEPs (see Table 3).

The optimal portfolios for S&P500/S&P500 ESG stock and bond investments are shown in Figure 2.

See Figure 2.

S&P500 index. It follows from Figures 2 (A) and (B) that when the S&P500 index is chosen, the impact of climate uncertainty on the stock investment strategy is significant: as climate uncertainty θ increases from 1 to 20, the stock demand decreases from 71% to 15%. The overall portfolio in a low climate uncertainty case ($\theta = 1$) is leveraged and requires the investor to borrow about 55% of initial wealth from the money market. This leverage is not surprising given the relatively good risk–return trade-off of the S&P500 and the possibility to hedge interest rate risk by trading in the bond. In particular, the Sharpe ratio for the index is 0.53 ($= (7.98 - 0.1)/14.91$, see also Section 5.1) which when coupled with rather small model uncertainty makes the stock an attractive investment. On the other hand, as the climate uncertainty becomes high ($\theta = 20$) the portfolio does not require any leverage. Although the speculative component for the bond changes with climate uncertainty, the bond hedge demand does not change (see the grey line in Figure 2 (B)).

The stock hedge demand is very small (grey lines in Figure 2(A)). Furthermore, investor’s hedge against the unobservable state predictor is also low. In this respect, we would like to note that both findings are rather standard in portfolio choice literature (see, for example, Branger et al. (2013)).

S&P500 ESG index. Similarly to S&P500 investment, S&P500 ESG holdings are also highly dependent on climate uncertainty (see Figure 2 (C)). For a low level of uncertainty ($\theta = 1$) an investor allocates 128% of wealth to the stock, whereas for a high level of uncertainty ($\theta = 4$) the proportion of the wealth allocated to the stock is 79% which is a 38% decrease in the stock demand due to climate uncertainty. For the bond investment (see Figure 2 (D)), both the speculative and hedge bond demands are affected by climate uncertainty. In general, the magnitudes of speculative and hedge components decrease with climate uncertainty. Similar to the S&P500 case, the stock hedge demand is fairly small.

5.5. *Climate scenarios*

As discussed in Section 5.3, climate uncertainty translates into plausible worst-case scenarios for temperature change and expected stock returns. To determine the worst-case scenarios, we employed statistical tools such as DEPs which, in turn, depend on the amount of available *past* information (i.e., depend on N (see also Table 3)). On the other hand, the past information, no matter how abundant, could be a poor representation of future tendencies and this is especially true for phenomena such as climate change. This is due to the high degree of uncertainty that stems from rather limited understanding of climate science and ambiguity regarding policy-makers’ responses and related effectiveness. Thus, it makes sense to consider scenarios that are derived from temperature projections typically given in climate science literature and to explore the corresponding portfolio implications. Our modelling framework is particularly suitable for such temperature scenario-based analysis of portfolio choice.

As global average temperature increases, relative to 2020, by the end of this century, we consider the following 3 possible temperature change scenarios: $2^\circ C$ (optimistic), $4^\circ C$ (medium), and $6^\circ C$ (pessimistic). These scenarios are considered in Daniel et al.

(2019) together with probabilities of each scenario under various mitigation policies.¹⁰ Each temperature projection corresponds to a different long-run level of mean reversion $\bar{\lambda}$ in the temperature change dynamics (5). In particular, $2^\circ C$, $4^\circ C$, and $6^\circ C$, correspond to $\bar{\lambda} = 0.025$ (i.e., $0.025 \times 80 = 2$), $\bar{\lambda} = 0.05$, and $\bar{\lambda} = 0.075$, respectively.

We consider the impact of predictive power p_λ of temperature change and speed of temperature change mean reversion κ_λ on S&P500 stock and bond portfolios under each climate scenario (see Figure 3).

See Figure 3.

We analyze the optimal portfolios in Figure 3 (A),(B) when the investment horizon is equal to 20 years. It can be observed from Figure 3 (A) that as the predictive power of the temperature change decreases stock and bond allocations decrease. This is due to the negative impact of temperature change on the stock expected return (see parameters in Table 1). As should be expected, more adverse temperature scenarios imply smaller investments. Furthermore, the impact of the predictive power is larger on stock investments under more extreme climate scenarios. For example, as the predictive power decreases from -1 to -4 , the stock allocation changes by only 3% in the optimistic scenario of $2^\circ C$ and by 30% in the pessimistic scenario of $6^\circ C$. The same holds for the bond portfolio but this effect is less pronounced.

The impact of temperature change speed of mean reversion is shown in Figure 3 (B). Although the change in mean reversion does not result in large portfolio changes, a decrease in speed decreases portfolio's stock and bond allocations. Intuitively, a smaller mean reversion implies larger variance of the temperature change process. This increased uncertainty in the stock return predictive variable makes investors to decrease their stock and bond allocations.

We also studied the impact of temperature change volatility σ_λ and stock-temperature change correlation $\rho_{S\lambda}$ on the optimal portfolios. Since the portfolios change only slightly when these parameters change, we do not provide any plots for this analysis. Similarly, we do not provide the plots for the S&P500 ESG index because the conclusions are similar to the S&P500 index case.

5.6. *Certainty equivalent costs*

In this section we analyze welfare losses from following suboptimal strategies discussed in Section 4. The loss from ignoring climate uncertainty is shown in Figure 4.

See Figure 4.

As follows from Figure 4 (A), (B), the loss from ignoring climate uncertainty can be large for both the S&P500 and S&P500 ESG indices. For example, when the investment horizon is 20 years the loss from ignoring climate uncertainty for S&P500 (S&P500 ESG) is 10% (40%) when the climate uncertainty is low ($\theta = 1$) and 80% (88%) when the climate uncertainty is high ($\theta = 20$ for S&P500 and $\theta = 4$ for S&P500 ESG). As should be expected, the loss decreases as the investment horizon decreases. However, even for a relatively short investment horizon of 5 years the loss can be as high as 35% for S&P500 and 40% for S&P500 ESG in the high uncertainty case.

The welfare losses from ignoring climate uncertainty for different climate scenarios are shown in Figure 4 (C), (D). We show the results only for the S&P500 index

¹⁰See Table S2 in Supplementary Information for Daniel et al. (2019) for a distribution of different temperature change scenarios.

because the results for S&P500 ESG index are similar. It follows from Figure 4 (C), (D) that the loss is smaller for more adverse climate scenarios. To understand why this happens, one has to notice that the stock and bond investments are lower for adverse climate scenarios (see Figure 3) and any losses on these smaller holdings are not as significant as losses on larger stock and bond allocations. Another observation is that the losses increase with the climate uncertainty that is being ignored. For instance, for the optimistic climate scenario the welfare loss from ignoring climate uncertainty is about 15% when uncertainty is low ($\theta = 1$, see Figure 4(C)) and is more than 80% when uncertainty is high ($\theta = 20$, see Figure 4(D)).

Next, we consider welfare loss due to the presence of climate uncertainty (see Figure 5).

See Figure 5.

In general, the conclusions are similar to the conclusions made from Figure 4 except for the fact that the losses are smaller. Nonetheless, the losses are significant in all cases. For example, the loss from presence of even low climate uncertainty ($\theta = 1$) are more than 10% of initial wealth for the investor in the S&P500 index with 20 years investment horizon. The corresponding loss for S&P500 ESG investment is even higher.

6. Conclusions

In this paper we have investigated the problem of optimal portfolio choice which explicitly accounts for climate uncertainty. We have derived the optimal stock–bond–cash portfolio and studied the impact of uncertainty about climate change for portfolios with different investment horizons. We have empirically examined the differences between two types of investment: the S&P500 index and the S&P500 ESG index.

Our analysis provides some insights for investors and policy makers. We have shown that climate uncertainty has a substantial impact on investment in both indices. Furthermore, we have demonstrated that a delay in climate policies implementation can be detrimental to an investor’s welfare. In addition, ignoring climate uncertainty when making investment decisions could yield large welfare losses, even for portfolios with relatively short investment horizons.

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Appendix A. Optimal Filtering

We rewrite the model as

$$\begin{bmatrix} \frac{dS_t}{S_t} \\ d\lambda_t \\ dr_t \\ d\beta_t \end{bmatrix} = \begin{bmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t + p_\beta \beta_t) \\ \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \\ \kappa_\beta(\bar{\beta} - \beta_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_S & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & -\sigma_r & 0 \\ 0 & 0 & 0 & \sigma_\beta \end{bmatrix} \begin{bmatrix} dW_t^S \\ dW_t^\lambda \\ dW_t^P \\ dW_t^\beta \end{bmatrix}. \quad (31)$$

In terms of a Wiener process $[W_t^{(1)}, W_t^{(2)}, W_t^{(3)}, W_t^{(4)}]^\top$ the system becomes

$$\begin{bmatrix} \frac{dS_t}{S_t} \\ d\lambda_t \\ dr_t \\ d\beta_t \end{bmatrix} = \begin{bmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t + p_\beta \beta_t) \\ \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \\ \kappa_\beta(\bar{\beta} - \beta_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_S & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & -\sigma_r & 0 \\ 0 & 0 & 0 & \sigma_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho_{S\lambda} & \hat{\rho}_\lambda & 0 & 0 \\ \rho_{SP} & \hat{\rho}_{\lambda P} & \hat{\rho}_P & 0 \\ \rho_{S\beta} & \hat{\rho}_{\lambda\beta} & \hat{\rho}_{P\beta} & \hat{\rho}_\beta \end{bmatrix} \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \\ dW_t^{(4)} \end{bmatrix} \quad (32)$$

where $\hat{\rho}_\lambda = \sqrt{1 - \rho_{S\lambda}^2}$, $\hat{\rho}_{\lambda P} = \frac{\rho_{\lambda P} - \rho_{S\lambda}\rho_{SP}}{\sqrt{1 - \rho_{S\lambda}^2}}$, $\hat{\rho}_P = \sqrt{1 - \rho_{SP}^2 - \hat{\rho}_{\lambda P}^2}$, $\hat{\rho}_{\lambda\beta} = \frac{\rho_{\lambda\beta} - \rho_{S\lambda}\rho_{S\beta}}{\sqrt{1 - \rho_{S\lambda}^2}}$, $\hat{\rho}_{P\beta} = \frac{\rho_{P\beta} - \rho_{SP}\rho_{S\beta} - \hat{\rho}_{\lambda P}\hat{\rho}_{\lambda\beta}}{\hat{\rho}_P}$, $\hat{\rho}_\beta = \sqrt{1 - \rho_{S\beta}^2 - \hat{\rho}_{\lambda\beta}^2 - \hat{\rho}_{P\beta}^2}$. Equivalently,

$$\begin{bmatrix} \frac{dS_t}{S_t} \\ d\lambda_t \\ dr_t \\ d\beta_t \end{bmatrix} = \begin{bmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t + p_\beta \beta_t) \\ \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \\ \kappa_\beta(\bar{\beta} - \beta_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_S & 0 & 0 & 0 \\ \sigma_\lambda \rho_{S\lambda} & \sigma_\lambda \hat{\rho}_\lambda & 0 & 0 \\ -\sigma_r \rho_{SP} & -\sigma_r \hat{\rho}_{\lambda P} & -\sigma_r \hat{\rho}_P & 0 \\ \sigma_\beta \rho_{S\beta} & \sigma_\beta \hat{\rho}_{\lambda\beta} & \sigma_\beta \hat{\rho}_{P\beta} & \sigma_\beta \hat{\rho}_\beta \end{bmatrix} \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \\ dW_t^{(4)} \end{bmatrix}. \quad (33)$$

We use the notation as in Liptser and Shiryaev (2001). We rewrite the equations for the observed processes S, λ , and r separately from the unobserved process β . Thus, we have the observed variables

$$\begin{bmatrix} \frac{dS_t}{S_t} \\ d\lambda_t \\ dr_t \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{pmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t) \\ \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \end{pmatrix}}_{A_0} + \underbrace{\begin{pmatrix} \sigma_S p_\beta \\ 0 \\ 0 \end{pmatrix}}_{A_1} \beta_t \\ \underbrace{0}_{B_1} dW_t^{(4)} + \underbrace{\begin{bmatrix} \sigma_S & 0 & 0 \\ \sigma_\lambda \rho_{S\lambda} & \sigma_\lambda \hat{\rho}_\lambda & 0 \\ -\sigma_r \rho_{SP} & -\sigma_r \hat{\rho}_{\lambda P} & -\sigma_r \hat{\rho}_P \end{bmatrix}}_{B_2} \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{bmatrix} \end{bmatrix} dt \quad (34)$$

and the unobserved variable

$$d\beta_t = \underbrace{(\kappa_\beta \bar{\beta})}_{a_0} + \underbrace{(-\kappa_\beta)}_{a_1} \beta_t dt + \underbrace{\sigma_\beta \hat{\rho}_\beta}_{b_1} dW_t^{(4)} + \underbrace{[\sigma_\beta \rho_{S\beta}, \sigma_\beta \hat{\rho}_{\lambda\beta}, \sigma_\beta \hat{\rho}_{P\beta}]}_{b_2} \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{bmatrix}. \quad (35)$$

We evaluate the matrix $B \circ B = B_1 B_1^\top + B_2 B_2^\top$. We have $B \circ B = B_2 B_2^\top$ and

$(B \circ B)^{-1} = (B_2 B_2^\top)^{-1} = (B_2^\top)^{-1} B_2^{-1}$ where

$$B_2^{-1} = \begin{bmatrix} \frac{1}{\sigma_S} & 0 & 0 \\ -\frac{\rho_{S\lambda}}{\sigma_S \hat{\rho}_\lambda} & \frac{1}{\sigma_\lambda \hat{\rho}_\lambda} & 0 \\ \frac{\rho_{S\lambda} \hat{\rho}_{\lambda P} - \hat{\rho}_\lambda \rho_{SP}}{\sigma_S \hat{\rho}_\lambda \hat{\rho}_P} & -\frac{\hat{\rho}_{\lambda P}}{\sigma_\lambda \hat{\rho}_\lambda \hat{\rho}_P} & -\frac{1}{\sigma_r \hat{\rho}_P} \end{bmatrix}. \quad (36)$$

Therefore, if the elements of B_2^{-1} are uniformly bounded, then the elements of $(B \circ B)^{-1}$ are also uniformly bounded. This implies that function σ_S should be uniformly positive ($\sigma_S \geq c > 0$).

Applying Theorem 12.7 in Liptser and Shiryaev (2001) we have that the process $\hat{\beta}_t = E[\beta_t \mid \mathcal{F}_t^{S,\lambda,r}]$ with the variance $m_t = E[(\beta_t - \hat{\beta}_t)^2 \mid \mathcal{F}_t^{S,\lambda,r}]$ satisfies

$$\begin{aligned} d\hat{\beta}_t &= \underbrace{(\kappa_\beta \bar{\beta})}_{a_0} + \underbrace{(-\kappa_\beta) \hat{\beta}_t}_{a_1} dt + \left[b_2 B_2^\top + m_t \underbrace{\begin{bmatrix} \sigma_S p_\beta & 0 & 0 \end{bmatrix}}_{A_1^\top} \right] \\ &\times \underbrace{\begin{bmatrix} \frac{1}{\sigma_S} & -\frac{\rho_{S\lambda}}{\sigma_S \hat{\rho}_\lambda} & \frac{\rho_{S\lambda} \hat{\rho}_{\lambda P} - \hat{\rho}_\lambda \rho_{SP}}{\sigma_S \hat{\rho}_\lambda \hat{\rho}_P} \\ 0 & \frac{1}{\sigma_\lambda \hat{\rho}_\lambda} & -\frac{\hat{\rho}_{\lambda P}}{\sigma_\lambda \hat{\rho}_\lambda \hat{\rho}_P} \\ 0 & 0 & -\frac{1}{\sigma_r \hat{\rho}_P} \end{bmatrix}}_{(B \circ B)^{-1}} \begin{bmatrix} \frac{1}{\sigma_S} & 0 & 0 \\ -\frac{\rho_{S\lambda}}{\sigma_S \hat{\rho}_\lambda} & \frac{1}{\sigma_\lambda \hat{\rho}_\lambda} & 0 \\ \frac{\rho_{S\lambda} \hat{\rho}_{\lambda P} - \hat{\rho}_\lambda \rho_{SP}}{\sigma_S \hat{\rho}_\lambda \hat{\rho}_P} & -\frac{\hat{\rho}_{\lambda P}}{\sigma_\lambda \hat{\rho}_\lambda \hat{\rho}_P} & -\frac{1}{\sigma_r \hat{\rho}_P} \end{bmatrix} \\ &\times \left\{ \begin{bmatrix} \frac{dS_t}{S_t} \\ d\lambda_t \\ dr_t \end{bmatrix} - \left[\underbrace{\begin{pmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t) \\ \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \end{pmatrix}}_{A_0} + \underbrace{\begin{pmatrix} \sigma_S p_\beta \\ 0 \\ 0 \end{pmatrix}}_{A_1} \hat{\beta}_t \right] dt \right\}. \end{aligned} \quad (37)$$

Taking into account that vector

$$B_2^{-1} \times \left\{ \begin{bmatrix} \frac{dS_t}{S_t} \\ d\lambda_t \\ dr_t \end{bmatrix} - \left[\begin{pmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t) \\ \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \end{pmatrix} + \begin{pmatrix} \sigma_S p_\beta \\ 0 \\ 0 \end{pmatrix} \hat{\beta}_t \right] dt \right\} \quad (38)$$

denoted as $[d\hat{W}_t^{(1)}, d\hat{W}_t^{(2)}, d\hat{W}_t^{(3)}]^\top$, is a Brownian motion relative to the filtration $\mathcal{F}_t^{S,\lambda,r}$, we obtain the following representation for the observed processes

$$\begin{aligned} \begin{bmatrix} \frac{dS_t}{S_t} \\ d\lambda_t \\ dr_t \end{bmatrix} &= \begin{bmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t + p_\beta \hat{\beta}_t) \\ \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \end{bmatrix} dt \\ &+ \begin{bmatrix} -\frac{\sigma_S}{\sigma_\lambda \rho_{S\lambda}} & 0 & 0 \\ -\frac{\sigma_\lambda \rho_{S\lambda}}{-\sigma_r \rho_{SP}} & -\frac{\sigma_\lambda \hat{\rho}_\lambda}{-\sigma_r \hat{\rho}_{\lambda P}} & -\frac{0}{-\sigma_r \hat{\rho}_P} \end{bmatrix} \begin{bmatrix} d\hat{W}_t^{(1)} \\ d\hat{W}_t^{(2)} \\ d\hat{W}_t^{(3)} \end{bmatrix}. \end{aligned} \quad (39)$$

On the other hand, we also obtain

$$\begin{aligned} &\left[b_2 B_2^\top + [m_t \sigma_S p_\beta, 0, 0] \right] \times \begin{bmatrix} \frac{1}{\sigma_S} & -\frac{\rho_{S\lambda}}{\sigma_S \hat{\rho}_\lambda} & \frac{\rho_{S\lambda} \hat{\rho}_{\lambda P} - \hat{\rho}_\lambda \rho_{SP}}{\sigma_S \hat{\rho}_\lambda \hat{\rho}_P} \\ 0 & \frac{1}{\sigma_\lambda \hat{\rho}_\lambda} & -\frac{\hat{\rho}_{\lambda P}}{\sigma_\lambda \hat{\rho}_\lambda \hat{\rho}_P} \\ 0 & 0 & -\frac{1}{\sigma_r \hat{\rho}_P} \end{bmatrix} \\ &= \underbrace{[\sigma_\beta \rho_{S\beta} + m_t p_\beta]}_{A_1(t)} \underbrace{[\sigma_\beta \hat{\rho}_{\lambda\beta} - \frac{m_t p_\beta \rho_{S\lambda}}{\hat{\rho}_\lambda}]}_{A_2(t)} \underbrace{[\sigma_\beta \hat{\rho}_{P\beta} + \frac{m_t p_\beta (\rho_{S\lambda} \hat{\rho}_{\lambda P} - \hat{\rho}_\lambda \rho_{SP})}{\hat{\rho}_\lambda \hat{\rho}_P}]}_{A_3(t)}. \end{aligned} \quad (40)$$

Thus, the dynamics of $\hat{\beta}$ can be written as

$$d\hat{\beta}_t = \kappa_\beta(\bar{\beta} - \hat{\beta}_t)dt + A_1(t)d\hat{W}_t^{(1)} + A_2(t)d\hat{W}_t^{(2)} + A_3(t)d\hat{W}_t^{(3)}. \quad (41)$$

According to Theorem 12.8 in Liptser and Shiryaev (2001) the variance m_t is given by

$$\frac{dm_t}{dt} = -2\kappa_\beta m_t + \underbrace{\sigma_\beta^2}_{\text{cob}} - A_1^2(t) - A_2^2(t) - A_3^2(t). \quad (42)$$

The variance m_t converges to a positive value m defined by the equation

$$-2\kappa_\beta m + \sigma_\beta^2 - A_1^2 - A_2^2 - A_3^2 = 0, \quad (43)$$

where

$$A_1 = \sigma_\beta \rho_{S\beta} + mp_\beta, \quad (44)$$

$$A_2 = \sigma_\beta \hat{\rho}_{\lambda\beta} - \frac{mp_\beta \rho_{S\lambda}}{\hat{\rho}_\lambda}, \quad (45)$$

$$A_3 = \sigma_\beta \hat{\rho}_{P\beta} + \frac{mp_\beta(\rho_{S\lambda}\hat{\rho}_{\lambda P} - \hat{\rho}_\lambda \rho_{SP})}{\hat{\rho}_\lambda \hat{\rho}_P}. \quad (46)$$

Therefore, the filtered equations are

$$\frac{dS_t}{S_t} = (r_t + \sigma_S(p + p_\lambda \lambda_t + p_\beta \hat{\beta}_t))dt + \sigma_S d\hat{W}_t^{(1)}, \quad (47)$$

$$d\lambda_t = \kappa_\lambda(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda \rho_{S\lambda} d\hat{W}_t^{(1)} + \sigma_\lambda \hat{\rho}_\lambda d\hat{W}_t^{(2)}, \quad (48)$$

$$dr_t = \kappa_r(\bar{r} - r_t)dt - \sigma_r \rho_{SP} d\hat{W}_t^{(1)} - \sigma_r \hat{\rho}_{\lambda P} d\hat{W}_t^{(2)} - \sigma_r \hat{\rho}_P d\hat{W}_t^{(3)}, \quad (49)$$

$$d\hat{\beta}_t = \kappa_\beta(\bar{\beta} - \hat{\beta}_t)dt + A_1 d\hat{W}_t^{(1)} + A_2 d\hat{W}_t^{(2)} + A_3 d\hat{W}_t^{(3)}. \quad (50)$$

From these equations we also obtain the following representation

$$dW_t^P = \rho_{SP} d\hat{W}_t^{(1)} + \hat{\rho}_{\lambda P} d\hat{W}_t^{(2)} + \hat{\rho}_P d\hat{W}_t^{(3)}. \quad (51)$$

Thus, the bond price dynamics is

$$\frac{dP_t}{P_t} = (r_t + q)dt + \sigma_P \rho_{SP} d\hat{W}_t^{(1)} + \sigma_P \hat{\rho}_{\lambda P} d\hat{W}_t^{(2)} + \sigma_P \hat{\rho}_P d\hat{W}_t^{(3)}. \quad (52)$$

The Wiener processes B_t and B_t^Y in equations (8) and (9), respectively, are given by $dB_{t,1} = d\hat{W}_t^{(1)}$, $dB_{t,2} = \frac{\hat{\rho}_{\lambda P}}{\sqrt{1-\rho_{SP}^2}} d\hat{W}_t^{(2)} + \frac{\hat{\rho}_P}{\sqrt{1-\rho_{SP}^2}} d\hat{W}_t^{(3)}$, and $dB_{t,1}^Y = d\hat{W}_t^{(1)}$, $dB_{t,2}^Y = d\hat{W}_t^{(2)}$, $dB_{t,3}^Y = d\hat{W}_t^{(3)}$.

Appendix B. Solution to HJB equation

In this section we solve the HJB equation (19). From the first-order conditions for u , we obtain

$$u^* = \eta \left((\Sigma^Y k)^\top v_y + x \phi^\top \Sigma \rho^\top k v_x \right). \quad (53)$$

Substituting u^* from (53) into the HJB equation (19) we obtain

$$\begin{aligned} v_t + \sup_\phi \left\{ \frac{1}{2} x^2 \phi^\top \Sigma \Sigma^\top \phi v_{xx} + x \left(\phi^\top (\mu - r) + r \right) v_x \right. \\ \left. + x \phi^\top \Sigma \rho^\top \Sigma^{Y\top} v_{xy} + \frac{1}{2} \text{Tr} \left(\Sigma^Y \Sigma^{Y\top} v_{yy} \right) + \mu^{Y\top} v_y \right. \\ \left. - \eta \phi^\top \Sigma \rho^\top k \left(\Sigma^Y k \right)^\top v_y v_x - \frac{1}{2} \eta \left(\phi^\top \Sigma \rho^\top k v_x \right)^2 - \frac{1}{2} \eta \left(\left(\Sigma^Y k \right)^\top v_y \right)^2 \right\} = 0 \end{aligned} \quad (54)$$

We use the ansatz

$$v(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma} (g(t, y))^\gamma \quad (55)$$

and the transformation (18) to rewrite the HJB equation (54) as

$$\begin{aligned} & g_t + \sup_{\phi} \left\{ -\frac{1-\gamma}{2} \phi^\top \Sigma \Sigma^\top \phi g + \frac{1-\gamma}{\gamma} (\phi^\top (\mu - r) + r) g + \frac{1}{2} \text{Tr} \left(\Sigma^Y \Sigma^{Y\top} g_{yy} \right) \right. \\ & + (1-\gamma) \phi^\top \Sigma \rho^\top \Sigma^{Y\top} g_y + \frac{\gamma-1}{2g} \text{Tr} \left(\Sigma^Y \Sigma^{Y\top} g_y g_y^\top \right) + \mu^{Y\top} g_y \\ & \left. - \theta \phi^\top \Sigma \rho^\top k \left(\Sigma^Y k \right)^\top g_y - \frac{1-\gamma}{2\gamma} \theta \left(\phi^\top \Sigma \rho^\top k \right)^2 g - \frac{\gamma\theta}{2(1-\gamma)g} \left(\left(\Sigma^Y k \right)^\top g_y \right)^2 \right\} = 0 \end{aligned} \quad (56)$$

where we also divided by $\frac{x^{1-\gamma}}{1-\gamma} \gamma (g(t, y))^{\gamma-1}$. The partial derivatives of function g are defined similarly to the partial derivatives of function v (see (20)). From the first-order conditions for ϕ we obtain the optimal investment strategy

$$\phi^* = \left(\Sigma \left[\gamma I + \theta \rho^\top k k^\top \rho \right] \Sigma^\top \right)^{-1} \left[(\mu - r) + \gamma \Sigma \rho^\top \left[I - \frac{\theta}{1-\gamma} k k^\top \right] \Sigma^{Y\top} \frac{g_y}{g} \right] \quad (57)$$

where I is the identity matrix of appropriate dimension.

Substituting (57) into the HJB equation (56), we find that

$$g(t, y) = \exp \left(a(t) + b^\top(t) y + \frac{1}{2} y^\top Q(t) y \right) \quad (58)$$

where function a , vector b , and matrix Q satisfy the following system of ordinary differential equations

$$\begin{aligned} \frac{da}{dt} & + \left(\bar{k} + \frac{1-\gamma}{\gamma} g_0 \right)^\top b(t) + b(t)^\top \left(\frac{1}{2} \Sigma^Y \Sigma^{Y\top} + \frac{\gamma-1}{2} N^{(1)} \right) b(t) \\ & + \frac{1}{2} \text{Tr} \left(\Sigma^Y \Sigma^{Y\top} Q(t) \right) + \frac{1-\gamma}{2\gamma^2} H_0 = 0, \\ \frac{db}{dt} & + \left(-K + \frac{1-\gamma}{\gamma} g_1 \right)^\top b(t) + Q(t) \left(\Sigma^Y \Sigma^{Y\top} + (\gamma-1) N^{(1)\top} \right) b(t) \\ & + Q(t) \left(\bar{k} + \frac{1-\gamma}{\gamma} g_0 \right) + \frac{1-\gamma}{2\gamma^2} H_1^\top + \frac{1-\gamma}{\gamma} \delta_1^\top = 0, \\ \frac{dQ}{dt} & + \left(-K + \frac{1-\gamma}{\gamma} g_1 \right)^\top Q(t) + Q(t) \left(-K + \frac{1-\gamma}{\gamma} g_1 \right) \\ & + Q(t) \left(\Sigma^Y \Sigma^{Y\top} + (\gamma-1) N^{(1)} \right) Q(t) + \frac{1-\gamma}{4\gamma^2} (H_2 + H_2^\top) = 0, \end{aligned} \quad (59)$$

with

$$\begin{aligned} N^{(1)} = & \Sigma^Y \left[I + \frac{\gamma\theta}{(\gamma-1)^2} k k^\top \right. \\ & \left. - \gamma \left[I - \frac{\theta}{1-\gamma} k k^\top \right] \rho \left[\gamma I + \theta \rho^\top k k^\top \rho \right]^{-1} \rho^\top \left[I - \frac{\theta}{1-\gamma} k k^\top \right] \right] \Sigma^{Y\top} \end{aligned} \quad (60)$$

$$\bar{k} = [\kappa_\lambda \bar{\lambda}, \kappa_r \bar{r}, \kappa_\beta \bar{\beta}]^\top, \quad K = \text{diag}(\kappa_\lambda, \kappa_r, \kappa_\beta), \quad \delta_1^\top = [0, 1, 0], \quad (61)$$

$$g_0 = \begin{bmatrix} n_{11}^{(3)} \sigma_{SP} + n_{21}^{(3)} q \\ n_{12}^{(3)} \sigma_{SP} + n_{22}^{(3)} q \\ n_{13}^{(3)} \sigma_{SP} + n_{23}^{(3)} q \end{bmatrix}, \quad g_1 = \begin{bmatrix} n_{11}^{(3)} \sigma_{SP\lambda} & 0 & n_{11}^{(3)} \sigma_{SP\beta} \\ n_{12}^{(3)} \sigma_{SP\lambda} & 0 & n_{12}^{(3)} \sigma_{SP\beta} \\ n_{13}^{(3)} \sigma_{SP\lambda} & 0 & n_{13}^{(3)} \sigma_{SP\beta} \end{bmatrix}, \quad (62)$$

$$N^{(3)} = \gamma(\Sigma^\top)^{-1} \left(\gamma I + \theta \rho^\top k k^\top \rho \right)^{-1} \rho^\top \left(I - \frac{\theta}{1-\gamma} k k^\top \right) \Sigma^{Y\top}, \quad (63)$$

$$H_0 = \sigma_{SP}(n_{11}^{(2)} \sigma_{SP} + n_{12}^{(2)} q) + q(n_{21}^{(2)} \sigma_{SP} + n_{22}^{(2)} q), \quad (64)$$

$$H_1^\top = \begin{bmatrix} \sigma_{SP\lambda}(2n_{11}^{(2)} \sigma_{SP} + q(n_{12}^{(2)} + n_{21}^{(2)})) \\ 0 \\ \sigma_{SP\beta}(2n_{11}^{(2)} \sigma_{SP} + q(n_{12}^{(2)} + n_{21}^{(2)})) \end{bmatrix}, \quad (65)$$

$$H_2 = \begin{bmatrix} 2n_{11}^{(2)}(\sigma_{SP\lambda})^2 & 0 & 2n_{11}^{(2)}\sigma_{SP\lambda}^2 p_{\lambda p_\beta} \\ 0 & 0 & 0 \\ 2n_{11}^{(2)}\sigma_{SP\lambda}^2 p_{\lambda p_\beta} & 0 & 2n_{11}^{(2)}(\sigma_{SP\beta})^2 \end{bmatrix}, \quad (66)$$

$$N^{(2)} = \left(\frac{1}{\gamma} \Sigma \left(\gamma I + \theta \rho^\top k k^\top \rho \right) \Sigma^\top \right)^{-1}. \quad (67)$$

Appendix C. Proof of Proposition 3

Plugging the suboptimal strategy (25), the transformation (18) and the function (26) into the equation (24), we verify that the function \tilde{v} given by (26) solves the equation provided that $\tilde{a}(t)$, $\tilde{b}(t)$, and $\tilde{Q}(t)$ satisfy the following ordinary differential equations

$$\begin{aligned} \frac{d\tilde{a}}{dt} &+ R_1^\top \tilde{b} + \tilde{b}^\top \left(\frac{1}{2} \Sigma^Y \Sigma^{Y\top} + \frac{\gamma-1}{2} F_1 \right) \tilde{b} - \text{Tr} \left(\frac{1}{2} \Sigma^Y \Sigma^{Y\top} \tilde{Q} \right) + \frac{1-\gamma}{\gamma^2} R_3 = 0, \\ \frac{d\tilde{b}}{dt} &+ R_2^\top \tilde{b} + \tilde{Q} \left(\Sigma^Y \Sigma^{Y\top} + (\gamma-1) F_1^\top \right) \tilde{b} + \tilde{Q} R_1 + \frac{1-\gamma}{\gamma^2} R_4^\top = 0, \\ \frac{d\tilde{Q}}{dt} &+ R_2^\top \tilde{Q} + \tilde{Q} R_2 + \tilde{Q} \left(\Sigma^Y \Sigma^{Y\top} + (\gamma-1) F_1 \right) \tilde{Q} + \frac{1-\gamma}{2\gamma^2} (R_5 + R_5^\top) = 0, \end{aligned} \quad (68)$$

where

$$R_1 = \bar{k} + F_3^\top f(t), \quad R_2 = F_3^\top F(t) - K, \quad R_3 = \gamma f^\top(t) D_1 - \frac{\gamma}{2} f^\top(t) F_2 f(t), \quad (69)$$

$$R_4 = \gamma \delta_1^\top - \gamma f^\top(t) F_2^\top F(t) + \gamma (D_1^\top F(t) + f^\top(t) D_2), \quad (70)$$

$$R_5 = 2\gamma F^\top(t) D_2 - \gamma F^\top(t) F_2 F(t), \quad F_1 = \Sigma^Y \left[I + \frac{\gamma}{(\gamma-1)^2} k k^\top \theta \right] \Sigma^{Y\top}, \quad (71)$$

$$F_2 = \Sigma \left[\gamma I + \theta \rho^\top k k^\top \rho \right] \Sigma^\top, \quad F_3 = \Sigma \rho^\top \left[(1-\gamma) I - \theta k k^\top \right] \Sigma^{Y\top}, \quad (72)$$

$$D_1 = [\sigma_{SP}, q]^\top, \quad D_2 = \begin{bmatrix} \sigma_{SP\lambda} & 0 & \sigma_{SP\beta} \\ 0 & 0 & 0 \end{bmatrix}. \quad (73)$$

Appendix D. Estimation

Hereby, we summarize the state space model used in this paper

$$\begin{bmatrix} \frac{dS_t}{S_t} \\ d\lambda_t \\ dr_t \\ d\beta_t \end{bmatrix} = \begin{bmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t + p_\beta \beta_t) \\ \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \\ \kappa_\beta(\bar{\beta} - \beta_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_S & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & -\sigma_r & 0 \\ 0 & 0 & 0 & \sigma_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho_{S\lambda} & \hat{\rho}_\lambda & 0 & 0 \\ \rho_{SP} & \hat{\rho}_{\lambda P} & \hat{\rho}_P & 0 \\ \rho_{S\beta} & \hat{\rho}_{\lambda\beta} & \hat{\rho}_{P\beta} & \hat{\rho}_\beta \end{bmatrix} \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \\ dW_t^{(4)} \end{bmatrix} \quad (74)$$

where the stock price S_t , the de-trend temperature anomalies λ_t and the t-bills r_t are observable. The unobservable variable β_t is the latent variable of the state model. Latent variables are defined as all endogenous and exogenous variables of the structural model that are not observed.

The discrete-time approximation of the dynamic model (74) is given by

$$\begin{bmatrix} r_{S,t+\Delta t} \\ \lambda_{t+\Delta t} - \lambda_t \\ r_{t+\Delta t} - r_t \\ \beta_{t+\Delta t} - \beta_t \end{bmatrix} = \begin{bmatrix} r_t + \sigma_S(p + p_\lambda \lambda_t + p_\beta \beta_t) \\ \kappa_\lambda(\bar{\lambda} - \lambda_t) \\ \kappa_r(\bar{r} - r_t) \\ \kappa_\beta(\bar{\beta} - \beta_t) \end{bmatrix} \Delta t + \begin{bmatrix} \sigma_S & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & -\sigma_r & 0 \\ 0 & 0 & 0 & \sigma_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho_{S\lambda} & \hat{\rho}_\lambda & 0 & 0 \\ \rho_{SP} & \hat{\rho}_{\lambda P} & \hat{\rho}_P & 0 \\ \rho_{S\beta} & \hat{\rho}_{\lambda\beta} & \hat{\rho}_{P\beta} & \hat{\rho}_\beta \end{bmatrix} \begin{bmatrix} \sqrt{\Delta t} Z_{t+\Delta t}^{(1)} \\ \sqrt{\Delta t} Z_{t+\Delta t}^{(2)} \\ \sqrt{\Delta t} Z_{t+\Delta t}^{(3)} \\ \sqrt{\Delta t} Z_{t+\Delta t}^{(4)} \end{bmatrix} \quad (75)$$

where Δt is the time discrepancy which measures the frequency of the observation.

We estimate the structural model using a simulated moments estimator (SME) introduced by [?]. The SME extends the generalized method of moments (GMM) estimator ([?]) to allow for latent variables in the non-linear structural model. The moment conditions for the unknown parameters of latent variables do not have analytic representations, hence the classic GMM method is not applicable.

For each state variable, its discrete-time process is given by

$$r_{S,t+\Delta t} = (r_t + \sigma_S(p + p_\lambda \lambda_t + p_\beta \beta_t)) \Delta t + \varepsilon_{t+\Delta t}^S \quad (76)$$

$$\lambda_{t+\Delta t} = \lambda_t + \kappa_\lambda(\bar{\lambda} - \lambda_t) \Delta t + \varepsilon_{t+\Delta t}^\lambda \quad (77)$$

$$r_{t+\Delta t} = r_t + \kappa_r(\bar{r} - r_t) \Delta t + \varepsilon_{t+\Delta t}^r \quad (78)$$

$$\beta_{t+\Delta t} = \beta_t + \kappa_\beta(\bar{\beta} - \beta_t) \Delta t + \varepsilon_{t+\Delta t}^\beta \quad (79)$$

where

$$\begin{aligned} \varepsilon_{t+\Delta t}^S &= \sigma_S \sqrt{\Delta t} Z_{t+\Delta t}^{(1)} \\ \varepsilon_{t+\Delta t}^\lambda &= \sigma_\lambda \rho_{S\lambda} \sqrt{\Delta t} Z_{t+\Delta t}^{(1)} + \sigma_\lambda \hat{\rho}_\lambda \sqrt{\Delta t} Z_{t+\Delta t}^{(2)} \\ \varepsilon_{t+\Delta t}^r &= \sigma_r \rho_{SP} \sqrt{\Delta t} Z_{t+\Delta t}^{(1)} + \sigma_r \hat{\rho}_{\lambda P} \sqrt{\Delta t} Z_{t+\Delta t}^{(2)} + \sigma_r \hat{\rho}_P \sqrt{\Delta t} Z_{t+\Delta t}^{(3)} \\ \varepsilon_{t+\Delta t}^\beta &= \sigma_\beta \rho_{S\beta} \sqrt{\Delta t} Z_{t+\Delta t}^{(1)} + \sigma_\beta \hat{\rho}_{\lambda\beta} \sqrt{\Delta t} Z_{t+\Delta t}^{(2)} + \sigma_\beta \hat{\rho}_{P\beta} \sqrt{\Delta t} Z_{t+\Delta t}^{(3)} + \sigma_\beta \hat{\rho}_\beta \sqrt{\Delta t} Z_{t+\Delta t}^{(4)} \end{aligned}$$

We use the SME method to estimate the vector of parameters

$$\theta = \{\sigma_S, p, p_\lambda, p_\beta, \kappa_\lambda, \bar{\lambda}, \sigma_\lambda, \rho_{S\lambda}, \kappa_r, \bar{r}, \sigma_r, \rho_{\lambda P}, \rho_{SP}, \kappa_\beta, \bar{\beta}, \sigma_\beta, \rho_{S\beta}, \rho_{\lambda\beta}, \rho_{P\beta}\}$$

We generate n process of β_t by taking independent draws from the distribution of β_t and computing recursively.

$$\beta_{t+\Delta t} = H\left(\beta_t, \varepsilon_{t+\Delta t}^\beta, \{\kappa_\beta, \bar{\beta}, \sigma_\beta\}\right)$$

where H denote the right hand side of (79). Define $\bar{g}(\theta)$ sample mean of the simulated moment condition

$$\bar{g}(\theta) = \frac{1}{T} \sum_{t=1}^T \frac{1}{n} \sum_{i=1}^n g(\theta, x_t, \beta_t^i) \quad (80)$$

where x_t is the observable time series data. β_t^i is the i^{th} path of the simulated latent variable at time t .

The SME estimator θ_{SME} is obtained by solving a quadratic problem

$$\theta_{SME} = \arg \min_{\theta} \bar{g}(\theta)' W \bar{g}(\theta) \quad (81)$$

where $W = S_T^{-1}$ is a positive definite and symmetric weighting matrix. As in the GMM case, the optimal weighting, satisfies

$$S_T = \sum_{j=-\infty}^{\infty} \mathbb{E} ([g_t(\theta) - \mathbb{E}(g_t(\theta))] [g_t(\theta) - \mathbb{E}(g_t(\theta))]')$$

The core moment conditions are as follows

$$\begin{aligned} \mathbb{E}_t \varepsilon_{t+\Delta t}^S &= 0 & \text{and} & & \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^S \right)^2 &= \sigma_S^2 \Delta t, \\ \mathbb{E}_t \varepsilon_{t+\Delta t}^\lambda &= 0 & \text{and} & & \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^\lambda \right)^2 &= \sigma_\lambda^2 \Delta t, \\ \mathbb{E}_t \varepsilon_{t+\Delta t}^r &= 0 & \text{and} & & \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^r \right)^2 &= \sigma_r^2 \Delta t, \\ \mathbb{E}_t \varepsilon_{t+\Delta t}^\beta &= 0 & \text{and} & & \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^\beta \right)^2 &= \sigma_\beta^2 \Delta t, \\ \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^S \varepsilon_{t+\Delta t}^\lambda \right) &= \sigma_S \sigma_\lambda \rho_{S\lambda} \Delta t \\ \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^S \varepsilon_{t+\Delta t}^r \right) &= \sigma_S \sigma_r \rho_{SP} \Delta t \\ \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^S \varepsilon_{t+\Delta t}^\beta \right) &= \sigma_S \sigma_\beta \rho_{S\beta} \Delta t \\ \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^\lambda \varepsilon_{t+\Delta t}^r \right) &= \sigma_\lambda \sigma_r (\rho_{S\lambda} \rho_{SP} + \hat{\rho}_\lambda \hat{\rho}_{\lambda P}) \Delta t \\ \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^\lambda \varepsilon_{t+\Delta t}^\beta \right) &= \sigma_\lambda \sigma_\beta (\rho_{S\lambda} \rho_{S\beta} + \hat{\rho}_\lambda \hat{\rho}_{\lambda\beta}) \Delta t \\ \mathbb{E}_t \left(\varepsilon_{t+\Delta t}^r \varepsilon_{t+\Delta t}^\beta \right) &= \sigma_r \sigma_\beta (\rho_{SP} \rho_{S\beta} + \hat{\rho}_{\lambda P} \hat{\rho}_{\lambda\beta} + \hat{\rho}_P \hat{\rho}_{P\beta}) \Delta t \end{aligned}$$

We generate p moments (with p larger than the number of parameters $p \geq 19$).

Appendix E. Detection-error probabilities

Define the conditional characteristic functions

$$\begin{aligned} f_1(\omega, t, N) &= E^\mathbb{P}[\exp(i\omega \xi_{u,N}) \mid \mathcal{F}_t^{S,Z,r}] = E^\mathbb{P}[\Xi_{u,N}^{i\omega} \mid \mathcal{F}_t^{S,Z,r}], \\ f_2(\omega, t, N) &= E^\mathbb{Q}[\exp(i\omega \xi_{u,N}) \mid \mathcal{F}_t^{S,Z,r}] = E^\mathbb{Q}[\Xi_{u,N}^{i\omega} \mid \mathcal{F}_t^{S,Z,r}] \\ &= E^\mathbb{P}[\exp(i\omega \xi_{u,N}) \exp(\xi_{u,N}) \mid \mathcal{F}_t^{S,Z,r}] = E^\mathbb{P}[\Xi_{u,N}^{i\omega+1} \mid \mathcal{F}_t^{S,Z,r}], \end{aligned}$$

where $i = \sqrt{-1}$, ω is the transform variable, and $\xi_{u,t} = \ln \Xi_{u,t}$.

Since the conditional characteristic functions are martingales, the Feynman–Kac theorem implies that functions f_1 and f_2 satisfy the following partial differential equation

$$\begin{aligned} \frac{\partial f}{\partial t} + \mu^{Y\top} \frac{\partial f}{\partial y} + \frac{1}{2} \Xi_{u,t}^2 (1 + k_1^2 + k_2^2) (u_t^*)^2 \frac{\partial^2 f}{\partial \Xi_{u,t}^2} + \frac{1}{2} \text{Tr} \left(\Sigma^Y \Sigma^{Y\top} \frac{\partial^2 f}{\partial y^2} \right) \\ + \Xi_{u,t} u_t^* B \frac{\partial^2 f}{\partial \Xi_{u,t} \partial y} = 0, \end{aligned} \quad (82)$$

but with different terminal conditions $f_1(\omega, N, N) = \Xi_{u,N}^{i\omega}$ and $f_2(\omega, N, N) = \Xi_{u,N}^{i\omega+1}$. From (23) we also have (for convenience of notation)

$$u^* = \theta(m_1(t) + m_2(t)y), \quad (83)$$

where

$$m_1(t) = k^\top \rho \left(\gamma I + \theta \rho^\top k k^\top \rho \right)^{-1} \Sigma^{-1} d_0 + \left(\frac{\gamma}{1-\gamma} \left(\Sigma^Y k \right)^\top + k^\top \rho \left(I + \frac{\theta}{\gamma} \rho^\top k k^\top \rho \right)^{-1} \rho^\top \left(I - \frac{\theta}{1-\gamma} k k^\top \right) \Sigma^{Y^\top} \right) b(t), \quad (84)$$

$$m_2(t) = k^\top \rho \left(\gamma I + \theta \rho^\top k k^\top \rho \right)^{-1} \Sigma^{-1} d_1 + \left(\frac{\gamma}{1-\gamma} \left(\Sigma^Y k \right)^\top + k^\top \rho \left(I + \frac{\theta}{\gamma} \rho^\top k k^\top \rho \right)^{-1} \rho^\top \left(I - \frac{\theta}{1-\gamma} k k^\top \right) \Sigma^{Y^\top} \right) Q(t) \quad (85)$$

with

$$d_0 = (\sigma_{SP}, q)^\top, \quad (86)$$

$$d_1 = \begin{pmatrix} \sigma_{SP\lambda} & 0 & \sigma_{SP\beta} \\ 0 & 0 & 0 \end{pmatrix}. \quad (87)$$

We solve the PDE (82) for f_1 because the solution for f_2 is analogous and follows from f_1 by replacing $i\omega$ with $i\omega + 1$. Let us look for a solution in form

$$f_1(\omega, t, N) = \Xi_{1,t}^{i\omega} \exp \left(a_{DEP}(t) + b_{DEP}^\top(t) y + \frac{1}{2} y^\top Q_{DEP}(t) y \right). \quad (88)$$

Substituting the trial solution (88) into (82) and dividing the resulting equation by (88) yields

$$\begin{aligned} \frac{da_{DEP}}{dt} &+ \left(\bar{k} + g_0^{DEP} \right)^\top b_{DEP}(t) + b_{DEP}^\top(t) \left(\frac{1}{2} \Sigma^Y \Sigma^{Y^\top} \right) b_{DEP}(t) \\ &+ \frac{1}{2} \text{Tr} \left(\Sigma^Y \Sigma^{Y^\top} Q_{DEP}(t) \right) + \frac{1}{2} H_0^{DEP} = 0, \quad a_{DEP}(N) = 0, \\ \frac{db_{DEP}}{dt} &+ \left(-K + g_1^{DEP} \right)^\top b_{DEP}(t) + Q_{DEP}(t) \left(\Sigma^Y \Sigma^{Y^\top} \right) b_{DEP}(t) \\ &+ Q_{DEP}(t) \left(\bar{k} + g_0^{DEP} \right) + \frac{1}{2} H_1^{DEP^\top} = 0, \quad b_{DEP}(N) = \mathbf{0}, \\ \frac{dQ_{DEP}}{dt} &+ \left(-K + g_1^{DEP} \right)^\top Q_{DEP}(t) + Q_{DEP}(t) \left(-K + g_1^{DEP} \right) \\ &+ Q_{DEP}(t) \left(\Sigma^Y \Sigma^{Y^\top} \right) Q_{DEP}(t) + \frac{1}{4} (H_2^{DEP} + H_2^{DEP^\top}) = \mathbf{0}, \quad Q_{DEP}(N) = \mathbf{0}, \end{aligned}$$

where

$$\begin{aligned} \bar{k} &= [\kappa_\lambda \bar{\lambda}, \kappa_r \bar{r}, \kappa_\beta \bar{\beta}]^\top, \quad K = \text{diag}(\kappa_\lambda, \kappa_r, \kappa_\beta), \quad g_0^{DEP} = i\omega \theta m_1(t) B^\top, \\ B &= \begin{bmatrix} \underbrace{-\sigma_\lambda(\rho_{S\lambda} + k_1 \hat{\rho}_\lambda)}_{B^{(1)}}, & \underbrace{\sigma_r(\rho_{SP} + k_1 \hat{\rho}_{\lambda P} + k_2 \hat{\rho}_P)}_{B^{(2)}}, & \underbrace{-(A_1 + k_1 A_2 + k_2 A_3)}_{B^{(3)}} \end{bmatrix}, \\ g_1^{DEP} &= i\omega \theta \begin{bmatrix} m_2^{(1)}(t) B^{(1)} & m_2^{(2)}(t) B^{(1)} & m_2^{(3)}(t) B^{(1)} \\ m_2^{(1)}(t) B^{(2)} & m_2^{(2)}(t) B^{(2)} & m_2^{(3)}(t) B^{(2)} \\ m_2^{(1)}(t) B^{(3)} & m_2^{(2)}(t) B^{(3)} & m_2^{(3)}(t) B^{(3)} \end{bmatrix}, \\ H_0^{DEP} &= i\omega(i\omega - 1)(1 + k_1^2 + k_2^2) \theta^2 m_1^2(t), \\ H_1^{DEP} &= 2i\omega(i\omega - 1)(1 + k_1^2 + k_2^2) \theta^2 m_1(t) m_2(t), \\ H_2^{DEP} &= 2i\omega(i\omega - 1)(1 + k_1^2 + k_2^2) \theta^2 m_2^\top(t) m_2(t). \end{aligned}$$

Therefore, the detection-error probability is

$$\varepsilon_N(\theta) = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \left(\text{Re} \left[\frac{f_2(\omega, 0, N)}{i\omega} \right] - \text{Re} \left[\frac{f_1(\omega, 0, N)}{i\omega} \right] \right) d\omega.$$

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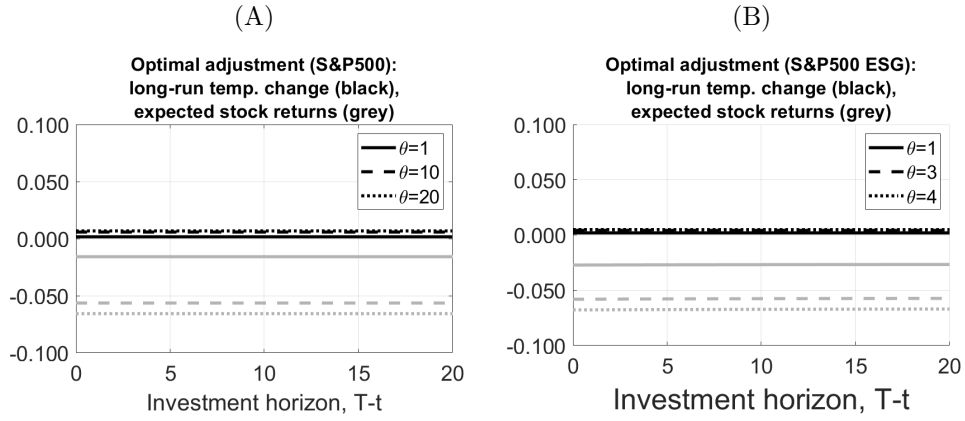


Figure 1. Optimal worst-case adjustments to expected stock returns and long-run temperature change as a function of investment horizon ($T-t$) and climate uncertainty (θ). Panels (A) shows the worst-case adjustments for the S&P500 stock returns; panel (B) shows the worst-case adjustments for the S&P500 ESG returns.

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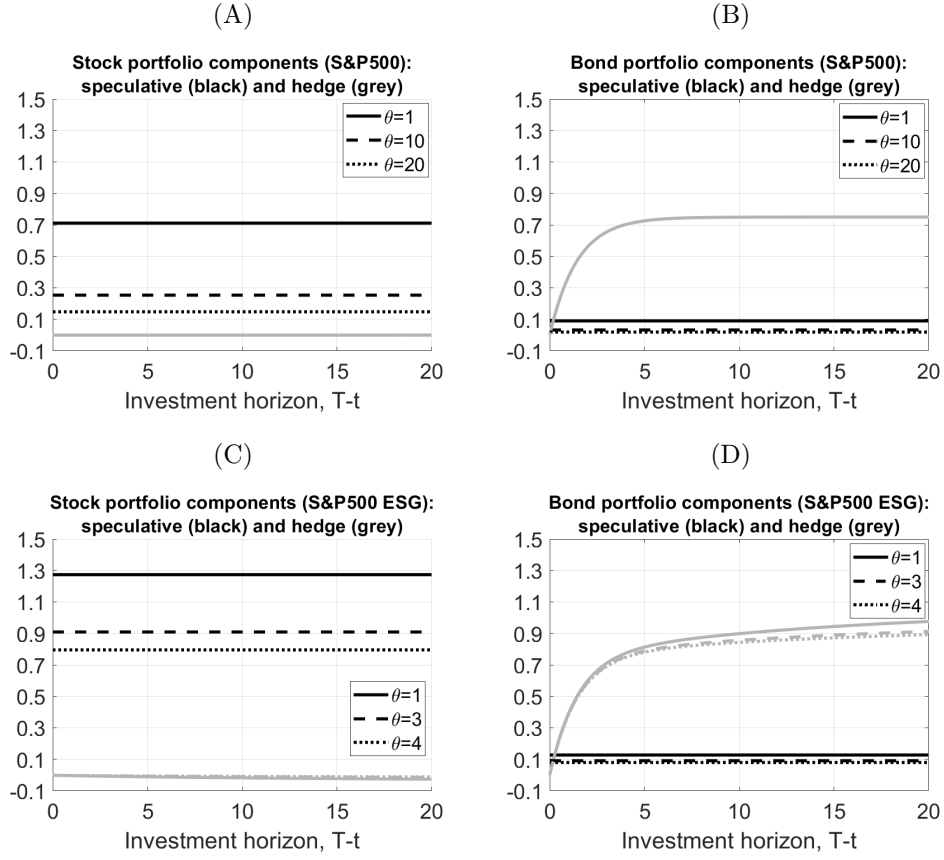


Figure 2. Optimal portfolio and its speculative and hedge components as a function of investment horizon ($T-t$) and climate uncertainty (θ). Panels (A) and (B) show portfolios when the stock index is represented by the S&P500; panels (C) and (D) show portfolios when the stock index is represented by the S&P500 ESG.

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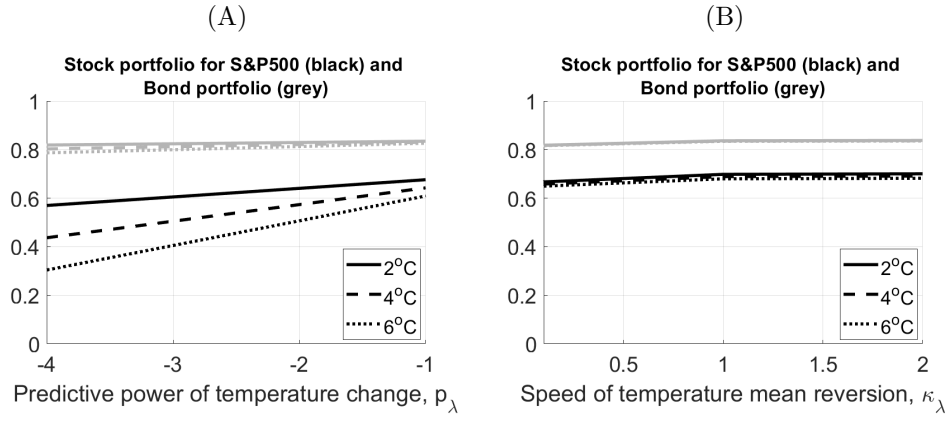


Figure 3. The impact of climate scenarios on optimal stock and bond portfolios as functions of predictive power of temperature change p_λ (panel (A)) and speed of temperature mean reversion κ_λ (panel (B)). The investment horizon is $T = 20$ years and the model uncertainty is $\theta = 1$.

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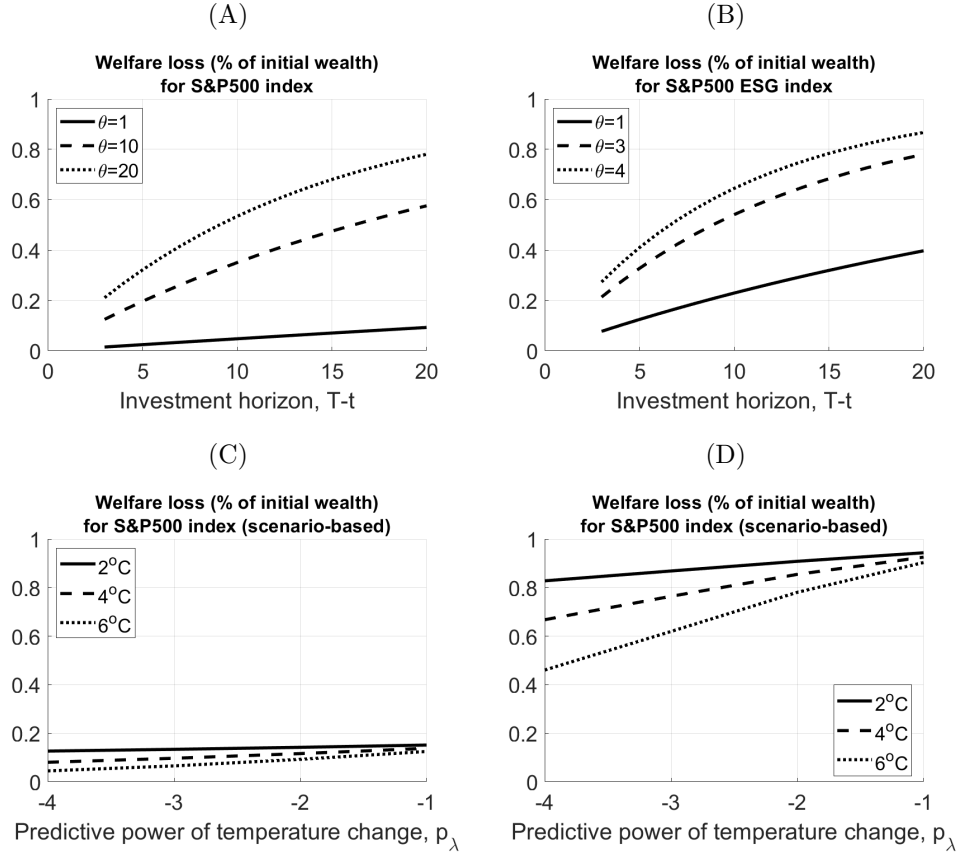


Figure 4. Welfare loss from ignoring climate uncertainty. Panel (A) shows the loss for S&P500 index; panel (B) shows the loss for the S&P500 ESG index; panel (C) shows the loss for S&P500 index as a function of predictive power of temperature change, p_λ , based on temperature scenarios (temperature increase by the end of the century relative to 2020) assuming that the model uncertainty is low ($\theta = 1$), the investment horizon is $T = 20$ years; panel (D) shows the loss for S&P500 index as a function of predictive power of temperature change, p_λ , based on temperature scenarios (temperature increase by the end of the century relative to 2020) assuming that the model uncertainty is high ($\theta = 20$), the investment horizon is $T = 20$ years.

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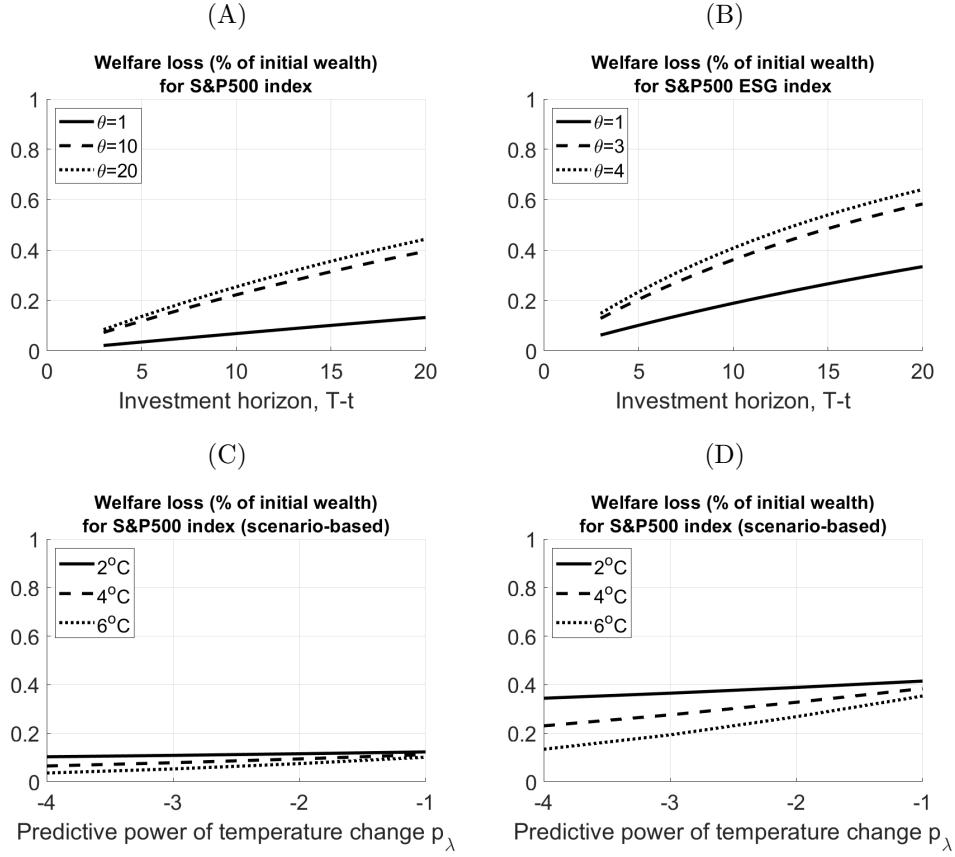


Figure 5. Welfare loss due to climate uncertainty. Panel (A) shows the loss for the S&P500 index; panel (B) shows the loss for the S&P500 ESG index; panel (C) shows the loss for the S&P500 index as a function of predictive power of temperature change, p_λ , based on temperature scenarios (temperature increase by the end of the century relative to 2020) assuming that the model uncertainty is low ($\theta = 1$), the investment horizon is $T = 20$ years; panel (D) shows the loss for S&P500 index as a function of predictive power of temperature change, p_λ , based on temperature scenarios (temperature increase by the end of the century relative to 2020) assuming that the model uncertainty is high ($\theta = 20$), the investment horizon is $T = 20$ years.

Table 1. Annualized parameter estimates for the S&P500 index (May 2005 – April 2021) and the state variables (based on monthly data). The estimates are estimated using Method of Simulated Moments (MSM) of Duffie and Singleton (1990).

| Parameter | Estimate |
|--|----------|
| <i>Temperature change: $d\lambda_t = \kappa_\lambda (\bar{\lambda} - \lambda_t) dt + \sigma_\lambda dW_t$</i> | |
| Mean reversion parameter, κ_λ | 7.1986 |
| Long-term level, $\bar{\lambda}$ | 0.0012 |
| Volatility parameter, σ_λ | 0.1136 |
| <i>Interest rate: $dr_t = \kappa_r (\bar{r} - r_t) - \sigma_r dW_t$</i> | |
| Mean reversion parameter, κ_r | 0.6880 |
| Long-term level, \bar{r} | 0.0010 |
| Volatility parameter, σ_r | 0.0017 |
| Correlation between λ and r , $\rho_{\lambda P}$ | -0.0651 |
| <i>Latent variable: $d\beta_t = \kappa_\beta (\bar{\beta} - \beta_t) dt + \sigma_\beta dW_t$</i> | |
| Mean reversion parameter, κ_β | 0.1625 |
| Long-term level, $\bar{\beta}$ | 0.0970 |
| Volatility parameter, σ_β | 0.0069 |
| <i>Stock returns: $\frac{dS_t}{S_t} = \left(r_t + \sigma_S (p + p_\lambda \lambda_t + p_\beta \beta_t) \right) dt + \sigma_S dW_t$</i> | |
| Volatility parameter, σ_S | 0.1491 |
| Market price of risk constant, p | 0.5145 |
| Predictive power of λ , p_λ | -0.2749 |
| Predictive power of β , p_β | 0.1462 |
| Correlation between $\frac{dS}{S}$ and λ , $\rho_{S\lambda}$ | -0.0124 |
| Correlation between $\frac{dS}{S}$ and r , ρ_{SP} | -0.0649 |
| <i>Correlations with latent variable:</i> | |
| Correlation between β and λ , $\rho_{\beta\lambda}$ | -0.0716 |
| Correlation between β and r , $\rho_{\beta P}$ | -0.0342 |
| Correlation between r_S and β , $\rho_{\beta S}$ | -0.0725 |

Table 2. Annualized parameter estimates for the S&P500 ESG index (May 2005–April 2021) and the state variables (based on monthly data). The estimates are estimated using Method of Simulated Moments (MSM) of Duffie and Singleton (1990).

| Parameter | Estimate |
|--|----------|
| <i>Temperature change: $d\lambda_t = \kappa_\lambda (\bar{\lambda} - \lambda_t) dt + \sigma_\lambda dW_t$</i> | |
| Mean reversion parameter, κ_λ | 7.1145 |
| Long-term level, $\bar{\lambda}$ | 0.0012 |
| Volatility parameter, σ_λ | 0.1127 |
| <i>Interest rate: $dr_t = \kappa_r (\bar{r} - r_t) - \sigma_r dW_t$</i> | |
| Mean reversion parameter, κ_r | 0.6912 |
| Long-term level, \bar{r} | 0.0013 |
| Volatility parameter, σ_r | 0.0017 |
| Correlation between λ and r , $\rho_{\lambda P}$ | -0.1196 |
| <i>Latent variable: $d\beta_t = \kappa_\beta (\bar{\beta} - \beta_t) dt + \sigma_\beta dW_t$</i> | |
| Mean reversion parameter, κ_β | 0.0646 |
| Long-term level, $\bar{\beta}$ | 0.5194 |
| Volatility parameter, σ_β | 0.0522 |
| <i>Stock returns: $\frac{dS_t}{S_t} = \left(r_t + \sigma_S (p + p_\lambda \lambda_t + p_\beta \beta_t) \right) dt + \sigma_S dW_t$</i> | |
| Volatility parameter, σ_S | 0.1458 |
| Market price of risk constant, p | 0.5249 |
| Predictive power of λ , p_λ | -0.5079 |
| Predictive power of β , p_β | 0.7744 |
| Correlation between $\frac{dS}{S}$ and λ , $\rho_{S\lambda}$ | -0.0448 |
| Correlation between $\frac{dS}{S}$ and r , ρ_{SP} | -0.0523 |
| <i>Correlations with latent variable:</i> | |
| Correlation between β and λ , $\rho_{\beta\lambda}$ | 0.1478 |
| Correlation between β and r , $\rho_{\beta P}$ | -0.2867 |
| Correlation between r_S and β , $\rho_{\beta S}$ | -0.1932 |

Table 3. Detection-error probabilities as a function of climate uncertainty (θ).

Panel (A): $N = 16$ years (baseline)

| S&P500 ESG index | | | | | |
|-----------------------------|--------|--------|--------|--------|--------|
| θ | 1 | 3 | 4 | 7 | 10 |
| $\varepsilon_N(\theta)$ | 0.2934 | 0.1385 | 0.1071 | 0.0640 | 0.0473 |

| S&P500 index | | | | | |
|-------------------------|--------|--------|--------|--------|--------|
| θ | 1 | 5 | 10 | 15 | 20 |
| $\varepsilon_N(\theta)$ | 0.3563 | 0.1796 | 0.1299 | 0.1112 | 0.1016 |

Panel (B): $N = 25$ years

| S&P500 ESG index | | | | | |
|-----------------------------|--------|--------|--------|--------|--------|
| θ | 1 | 3 | 4 | 7 | 10 |
| $\varepsilon_N(\theta)$ | 0.2548 | 0.0934 | 0.0657 | 0.0323 | 0.0211 |

| S&P500 index | | | | | |
|-------------------------|--------|--------|--------|--------|--------|
| θ | 1 | 5 | 10 | 15 | 20 |
| $\varepsilon_N(\theta)$ | 0.3597 | 0.1602 | 0.1010 | 0.0795 | 0.0686 |

Panel (C): $N = 40$ years

| S&P500 ESG index | | | | | |
|-----------------------------|--------|--------|--------|--------|--------|
| θ | 1 | 3 | 4 | 7 | 10 |
| $\varepsilon_N(\theta)$ | 0.2028 | 0.0479 | 0.0284 | 0.0098 | 0.0051 |

| S&P500 index | | | | | |
|-------------------------|--------|--------|--------|--------|--------|
| θ | 1 | 5 | 10 | 15 | 20 |
| $\varepsilon_N(\theta)$ | 0.3248 | 0.1044 | 0.0533 | 0.0374 | 0.0301 |