Systemic Risk Measures & Portfolio Choice

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SYSTEMIC RISK

EXECUTIVE SUMMARY

As opposed to a firm's individual risk of failure, which can be contained without harming the entire financial system, *systemic risk* is the risk of collapse of the entire financial system or market. Since the 2007–2009 financial crisis numerous attempts have been made to identify and measure the systemic risk of financial institutions. In this respect the following question arises: how can a given systemic risk measure be used to construct portfolios that perform relatively well when systemic risk materializes? In this paper we develop a framework for the optimal portfolio choice based on an exogenous systemic risk measure.

In his pioneering work on portfolio choice, Markowitz developed a theory of portfolio selection based on the "risk—return" characteristics of stocks in the portfolio. Markowitz's investor was assumed to be minimizing the variance (risk) of a portfolio's returns subject to meeting a given level of expected returns. In other words, Markowitz answered the question:

> *What portfolio of stocks will deliver a specified expected return and at the same time have the lowest variance of future returns?*

In this current research, we focus on adverse return scenarios and attempt to answer the question:

> *What portfolio of stocks will deliver the highest expected returns in the case of a financial crisis?*

A "financial crisis", or systemic event, is defined as a prolonged market decline. For the purpose of this paper a systemic event is defined as the drop in returns of a broad market index below their 5% VaR[1] level over a 6-months period (Figure 1).
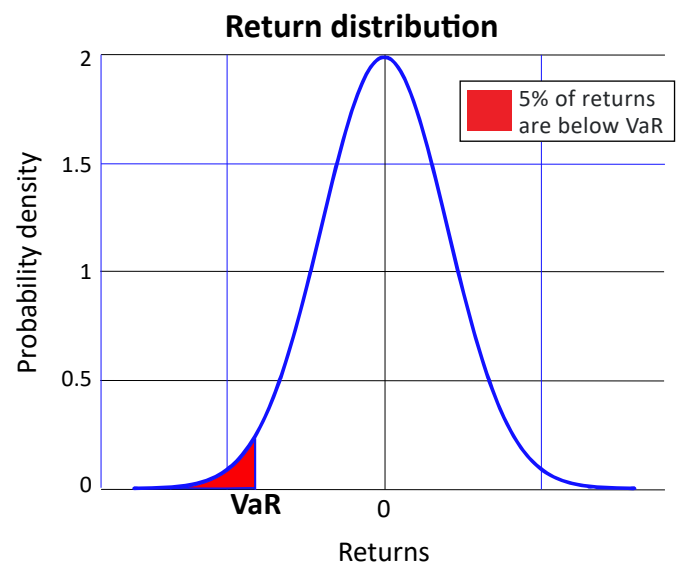

Figure 1: 5% VaR allows us to make statements of the form: We are 95% certain that the returns will not be less than VaR over, say, the next 6 months.

In addition to stressed market conditions, we consider only those portfolio's returns that are below the so-called, Conditional VaR (CoVaR), which is defined as the VaR of the portfolio's returns given that the market is in a crisis situation (market returns are below VaR). In this sense, 5% CoVaR allows us to make statements of the form: We are 95% certain that portfolio returns will not be less than

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1   Recall, Value-at-Risk estimates how much a set of investments might lose over a certain time period.

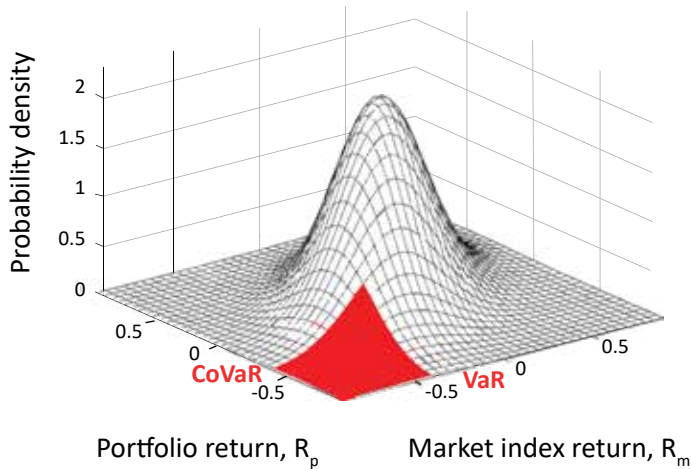CoVaR if market returns fall below VaR over, say, the next 6 months (Figure 2).

Figure 2: *The segment of market and portfolio's returns distribution that correspond to stressed market and portfolio scenarios. The segment can be viewed as a two-dimensional analogy to the more commonly used "tail of return distribution".*

The goal of our investor is to construct a portfolio that delivers maximal expected returns in the stressed market and portfolio's return scenarios (the red segment in Figure 2).

In the numerical part of our analysis we use five banks and three insurance companies: TD Bank, CIBC, RBC, Scotiabank, BMO, Manulife, Great West Lifeco, and Sunlife. As a benchmark portfolio we use the tangency portfolio, that is, the portfolio on the Markowitz efficient frontier that has the highest expected return per unit risk (standard deviation). To avoid large negative portfolio positions during stressed market conditions, we preclude short sales. We compare the benchmark and developed portfolio construction methodology by backtesting the portfolios on daily data that covers the period 2007-2017. The portfolios' performance is shown in Figure 3.

There are several notable features in Figure 3. First, both portfolios perform poorly during the 2007-2009 financial crisis and lose almost 50% of their value. The values of both portfolios significantly decline during this period because all stocks that we consider substantially lose in value. Second, and most importantly, our methodology has a superior performance during 2009-2012 European Sovereign Debt crisis when compared with the benchmark portfolio that loses almost 50%. Similarly, the benchmark portfolio value declines in the beginning of 2016 due to the declining price of oil, concerns regarding China's economic slowdown, and a weaker Canadian dollar. On the other hand, the CoVaR portfolio value is fairly stable during this period. Third, the Markowitz tangency portfolio has higher volatility than the CoVaR portfolio.
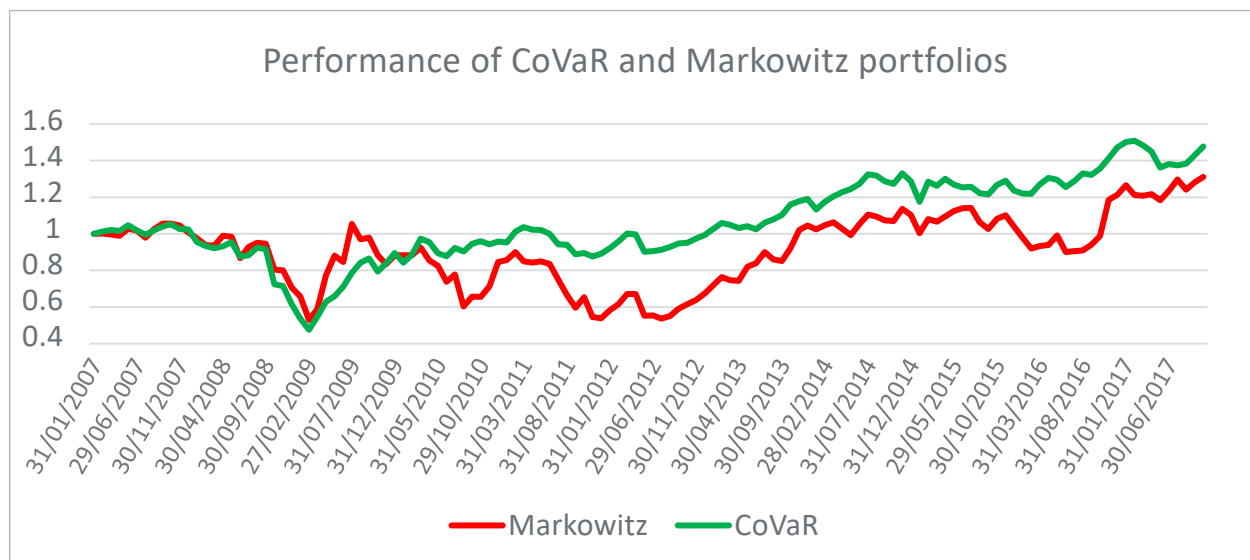

Figure 3: *The performance of the Markowitz tangency portfolio and the developed portfolio choice methodology (CoVaR) based on the initial investment of $1 on January 31, 2007. The y-axis shows the USD value of the portfolios.*
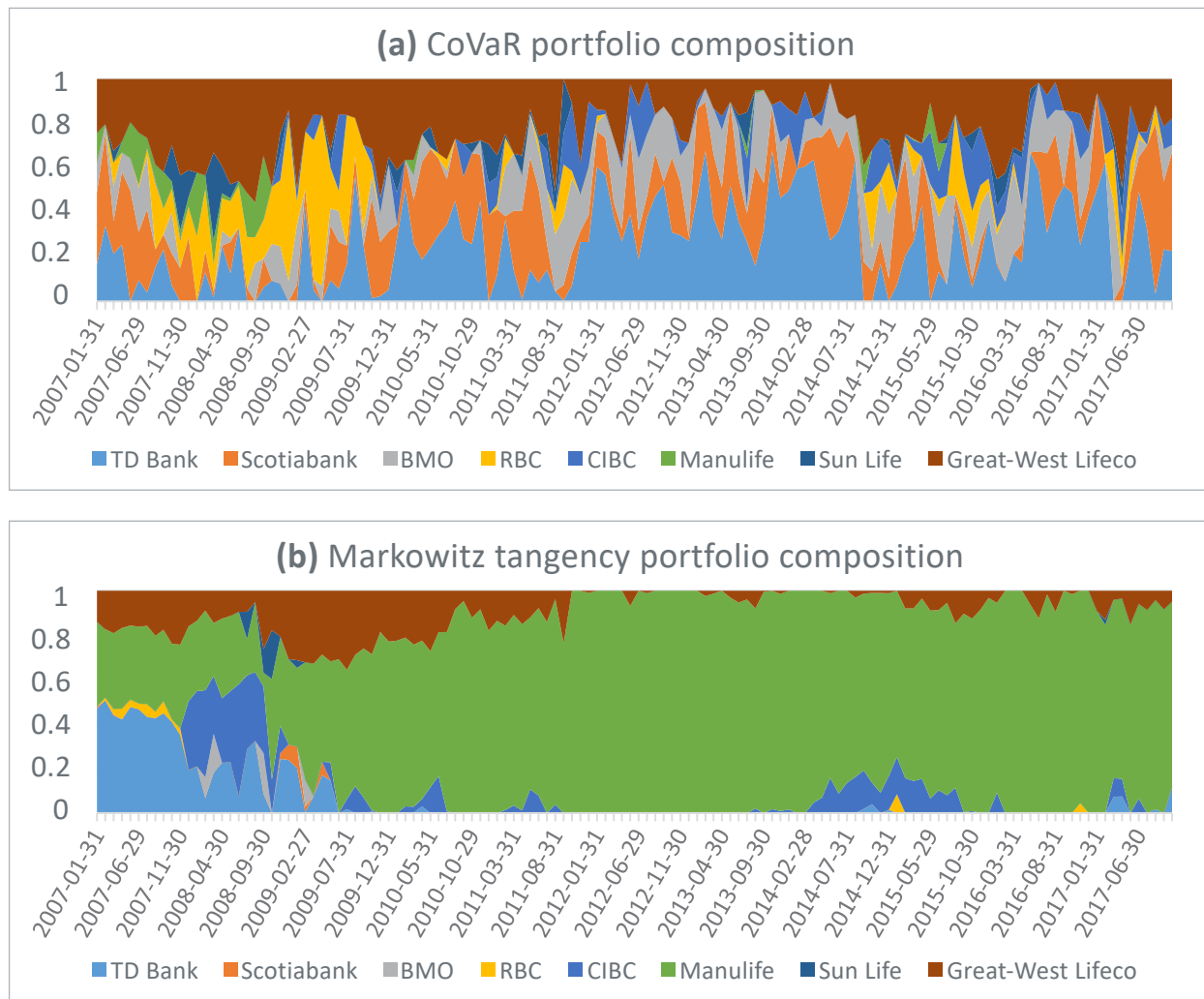
Figure 4: Composition of (a) CoVaR and (b) Markowitz tangency portfolios.

Figure 4 shows the portfolio constituents for the Markowitz tangency and CoVaR portfolios.

Importantly, from Figure 4 it follows that the CoVaR portfolio is more diversified than the Markowitz portfolio for which Manulife is the stock with the highest value of investment concentration starting from 2009. On the other hand, CoVaR portfolios imply relatively high investment in TD Bank and Great-West Lifeco which have been well known to be among the most consistent performers.

One of the most significant outcomes of the Markowitz portfolio analysis is the so-called "mutual fund separation theorem" which states that any portfolio on the Markowitz efficient frontier can be replicated by any two portfolios on the efficient frontier. This result implies that an investor can achieve a desired "risk-return" trade-off on the efficient frontier by trading in only two mutual funds, thereby reducing the transaction costs. In our research we show an extension of the mutual fund theorem, which requires three mutual funds. This result is due to the fact that in addition to risk and return characteristics of the portfolio, our approach also includes portfolio's correlation with the market, and therefore, more portfolios are required to obtain a desired "risk-return-correlation" trade-off.

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SYSTEMIC RISK MEASURES AND PORTFOLIO CHOICE

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Abstract. We develop a model for the optimal portfolio choice in presence of systemic risk. In our modelling approach we use two risk measures: VaR and CoVaR (Adrian and Brunnermeir (2011)). Our investor maximizes portfolio's expected returns conditioned on a systemic risk index being at (at most at) its VaR level and portfolio's returns below their CoVaR level. Under some assumptions the optimal investment strategy is derived in closed form. The proposed methodology is flexible in balancing the significance of portfolio's variance and portfolio's correlation with systemic risk index. The model is applied to the Canadian equity market and performs relatively well at the times of market downturns. Our approach also allows for ranking of financial institutions based on their systemic risk–return characterictics.

1. Introduction

As opposed to a firm's individual risk of failure, which can be contained without harming the entire financial system, *systemic risk* is the risk of collapse of the entire financial system or market. Since the 2007–2009 financial crisis numerous attempts have been made to identify and measure systemic risk of financial institutions (see, for example, Adrian and Brunnermeier (2011), Brunnermeier and Cheridito (2014), Brownlees and Engle (2016), Acharya et al. (2017)). In this respect the following question arises: how can a given systemic risk measure be used to construct portfolios that perform relatively well when systemic risk materializes? In this paper we develop a framework for the optimal portfolio choice based on exogenously given systemic risk measure.

Following the general framework of Adrian and Brunnermeier (2011), we consider portfolio returns conditioned on a systemic event which can be described as a precipitous deterioration of broad market conditions. In particular in our analysis we assume that there exists a systemic risk index that reflects the broad market conditions and that investors would like to take into account when making their portfolio decisions. On a technical level, we simultaneously work with two measures: VaR and CoVaR. VaR of the systemic risk index is defined as the most adverse change of the index at some prespecified level of confidence; CoVaR is the VaR of the portfolio conditioned on the systemic risk index being at (or below) its VaR level. The goal of our investor is to maximize the expected portfolio returns conditioned on (i) the systemic risk index being at (or below) its VaR level *and* (ii) the portfolio returns being below their CoVaR level. In simple terms, we seek the portfolio that performs best in a low return environment and when the overall market is in distress.

Althought there are different ways to model systemic events (see Brownlees and Engle (2016) and Adrian and Brunnermeier (2011), for example), we consider two formulations: (1) when the systemic risk index is *at* its VaR level and (2) when the index is *at most at* its VaR level. We obtain an explicit solution to our portfolio choice problem under formulation (1) and, by analyzing its properties, we highlight the main economic intuition for the investor's optimal behavior under the (more complex) formulation (2). For example, for formulation (1) the quantile parameters, that determine the values of VaR and CoVaR, allow our investor to properly balance the relative importance of portfolio's variance and portfolio's correlation with the systemic risk index. This intuition also holds for formulation (2).

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Our work advances existing literature on portfolio choice because it is the first to account for the role of systemic risk measures in optimal portfolio decisions. There are several contributions in our efforts. First, under the joint normality assumption we obtain a closed-form solution of the optimal portfolio weights when systemic risk index is set at the portfolio VaR level. We obtain economic interpretation for the quantile parameters: VaR quantile controls the importance of portfolio correlation with the index, whereas CoVaR quantile allows the investor to manage the significance of portfolio variance. Furthermore, we establish the following mutual fund separation result: any optimal portfolio with given expected return, variance, and covariance with systemic risk index can be replicated by only *three* appropriately chosen optimal portfolios. In addition, we show that when the assets in the portfolio are uncorrelated with systemic risk index, the optimal portfolio is mean–variance efficient.

Second, for the formulation when systemic risk index is at most at its VaR level we derive a closed-form expression for the investor's objective function and show that it is decreasing in portfolio standard deviation as well as correlation with the index. We also find that the higher is the portfolio's standard deviation, and the larger is the role that portfolio's correlation with systemic risk index plays in investor's portfolio choice. An intuitive explanation for this result is that when the variance of the portfolio is large, the downside risk is large if the portfolio is highly correlated with systemic risk index (since returns are bounded by CoVaR, negative portfolio returns become more likely). Similarly to the other formulation, we show that the optimal portfolio becomes mean–variance efficient when the assets are not correlated with the index.

Finally, we apply the developed model to the Canadian financial services industry (five banks and three insurance companies) using Bloomberg daily data from January 3, 2000 to October 31, 2017. To model the joint return dynamics and to generate scenarios of future returns we use GARCH Dynamic Conditional Correlation (GARCH–DCC) model (see Engle (2002, 2009)). Backtesting our methodology starting from January 2006, we find that at the times of market downturns our approach allows to avoid significant losses during the times of market downturns as compared with the classical tangency portfolio that we use as a benchmark. The only exception to this result is the 2007–2009 financial crisis when prices of all stocks significantly dropped making both portfolios almost equally lose in value.

A brief review of existing literature on multivariate portfolio choice is in order. Our modelling approach is related to the tail risk measures studied by Alexander and Baptista (2004) who study portfoio selection with VaR and CVaR constraints (see also Rockafellar and Uryasev (2002)). However, Alexander and Baptista (2004) consider only portfolio returns and do not consider systemic risk. Ang and Bekaert (2002) numerically solve the asset allocation problem when there are two switching regimes in the economy with one regime having higher volatilities and correlations. Das and Uppal (2004) model co-movement in asset returns by introducing Poisson jumps that arrive at the same time in a setting with a constant opportunity set. Buraschi et al. (2010) and Bäuerle and Li (2013) study intertemporal portfolio choice when the stochastic covariance matrix of asset returns is modelled by a Wishart diffusion process and Da Fonseca et al. (2011) extend the framework to the complete market setting by allowing the investor to trade in variance swaps. A more practical approach to portfolio construction in presence of systemic risk was also studied in Biglova et al. (2014).

This paper is organized as follows. In Section 2 we discuss the assumptions made in our modelling and provide some common notation. In Section 3 we solve the portfolio choice problem when the systemic risk measure is assumed to be at its VaR level. In Section 4 we characterize the portfolio choice problem for more adverse scenarious of the systemic risk measure. In Section 5 we empirically test our model on the Canadian equity market. Section 6 concludes.

2. Assumptions and Notation

In this section we state some important assumptions made in our modelling. We also provide some general notation that will be used throughout the paper.

Although different definitions of systemic event can be adopted, in what follows we define it as a severe market decline or, equivalently, a precipitous drop in a market index. This definition is similar to that of Brownlees and Engle (2016), for example, who assume that the triggering systemic event of the financial crisis is a decline (40% drop over 6 months) in a market index. Thus, in what follows we refer to the systemic risk index as the *market index*. It is on purpose that in our analysis we avoid the use of any specific risk measure as there is still an ongoing debate on how to measure systemic risk; we stay away from this

discussion because it is not directly related to this research.[1] In this respect, we would like to emphasize that our model is fairly general in that any systemic risk index can be used to model systemic risk and in this sense our approach can be regarded as a framework for portfolio choice in presence of systemic risk.

We assume that there is no risk-free asset and there are $n \geq 2$ risky assets with stochastic rates of return $r = (r_1, ..., r_n)^\mathsf{T}$. The expected rates of return are denoted as $\mu = \mathbb{E}[r]$ and $\Sigma = \mathbb{E}\left[(r - \mu)(r - \mu)^\mathsf{T}\right]$ represents the variance–covariance matrix of rates of return of the risky assets. The proportions of wealth invested in each of the assets are given by $w = (w_1, ..., w_n)^\mathsf{T}$ (w_i is the proportion of wealth invested in asset i) implying that $\sum_{i=1}^{n} w_i = 1$. Also we let $R_p = w^\mathsf{T} r$ and $\mu_p = w^\mathsf{T} \mu$ be the portfolio's rate of return and the portfolio's expected rate of return, respectively. Based on this notation, the portfolio returns' variance is given by $\sigma_p^2 = w^\mathsf{T} \Sigma w$. We also denote by μ_m the expected return and by σ_m the standard deviation of the returns on the market index.

To gain a better understanding of the systemic risk component that is taken into account in our portfolio choice problem, we assume that the returns on the assets in the portfolio and the market index are jointly normal. This assumption allows us to obtain some analytic results and to compare with other well-known portfolio choice approaches.

3. Portfolio Selection: Market is at its VaR level

In this section we consider a portfolio choice problem that assumes that the market index is at its VaR level and the portfolio returns are below their CoVaR levels. In this regard, we review VaR and CoVaR measures in Section and in Section 3.2 we formulate and solve the portfolio choice problem and discuss some intuitive aspects of our model.

3.1. **Exposure CoVaR of Adrian and Brunnermeier (2011).** In this section we review the systemic risk measure, exposure CoVaR, introduced by Adrian and Brunnermeier

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[1] On December 18, 2014, one the largest US insurance companies, MetLife, was notified by the Financial Stability Oversight Council (FSOC) that it had been designated a non-bank Systemically Important Financial Institution (SIFI). MetLife challenged that decision in federal court and on March 30, 2016 U.S. District Court Judge Rosemary Collyer ruled in MetLifes favor and rescinded FSOCs designation of the company as a SIFI. The Department of Justice on behalf of FSOC has appealed that decision and the case is now under consideration with the U.S. Court of Appeals for the DC Circuit. See https://www.metlife.com/sifiupdate/index.html

(2011). To simplify the exposition, we will refer to "exposure CoVaR" as CoVaR in all subsequent analysis.[2] The market index's VaR is defined as the value VaR_{q_m} such that

$$\mathbb{P}\left(R_m \le VaR_{q_m}\right) = q_m \qquad (3.1)$$

where R_m is the return on the market index that represents the state of a financial system. The quantile level q_m is typically chosen to be 0.1, 0.05, or 0.01. Building on the above definition of VaR, CoVaR of a portfolio, $CoVaR_{q_p}$, is defined as

$$\mathbb{P}\left(R_p \le CoVaR_{q_p}\Big|R_m = VaR_{q_m}\right) = q_p \qquad (3.2)$$

where R_p is the return on the portfolio. In other words, CoVaR is defined as the VaR of portfolio conditional on the market being in distress. In this sense CoVaR addresses the question *what portfolios are most exposed to a financial crisis.*[3]

From the investor's perspective, it is of interest to find a portfolio that performs reasonably well not only in good times, but also when the entire financial system is in a downturn. In this respect, portfolios that deliver a relatively high level of return and behave well in crises are highly desirable. To somehow immunize a given portfolio against market downturns, systemic risk should be directly incorporated in the portfolio optimization procedure.

3.2. **Portfolio choice.** To formulate the portfolio choice problem in presence of systemic risk, we consider the co-expected returns defined as

$$CoER^{=} = \mathbb{E}\left[R_p\Big|R_p \le CoVaR_{q_p}, R_m = VaR_{q_m}\right]. \qquad (3.3)$$

We would like to emphasize that in (3.3) we condition on both $\left\{R_p \le CoVaR_{q_p}\right\}$ and $\left\{R_m = VaR_{q_m}\right\}$. The portfolio returns for the specification (3.3) are graphically illustrated in Figure 1.

In other words, $CoER^{=}$ estimates the expected returns in a low return environment when the overall market is in distress (market is *at* its VaR level). Thus, the portfolio

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[2]CoVaR and "exposure CoVaR" of Adrian and Brunnermeier (2011) differ only on the conditioned event. In this respect, CoVaR is defined as a quantile of the market return distribution conditioned on ith institution's returns being at their VaR level, whereas "exposure CoVaR" is a quantile of ith institution return distribution conditioned on market returns being at their VaR level.

[3]Another systemic risk measure, SRISK of Brownlees and Engle (2016), is similar to "exposure CoVaR" in this sense: in SRISK calculation the conditioning is on a crisis which is defined as a 40% decrease of the market index.
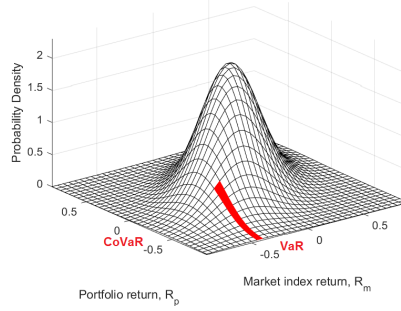
FIGURE 1. Illustration of the returns on which we condition in $CoER^=$. For a given probability distribution of returns on the market index and portfolio (in grey) we consider only those returns that correspond to stressed scenarios (in red): portfolio returns are below CoVaR ($R_p \leq CoVaR_{q_p}$) and market returns are at their VaR level ($R_m = VaR_{q_m}$).

choice problem can be stated as

$$\max_{w} \quad CoER^= \tag{P1}$$

$$\text{s.t.} \quad w^\mathsf{T}\mathbf{1} = 1, \tag{3.4}$$

where $\mathbf{1} = (1, ..., 1)^\mathsf{T} \in \mathbb{R}^n$. By solving (**P1**) we find the portfolio that performs relatively well when the market is at its VaR level and portfolio's returns are below CoVaR.

To solve the portfolio choice problem (**P1**), we first obtain a closed-form expression for $CoER^=$ which allows for a more in-depth analysis of the measure.

Lemma 1. *Assume that $(R_p, R_m) \sim \mathcal{BN}(\mu_p, \mu_m, \sigma_p, \sigma_m, \rho)$ where μ_p (μ_m) is the expected portfolio (market) return, σ_p (σ_m) is portfolio (market) return standard deviation, and $\rho \geq 0$ is the correlation between the returns on the portfolio and the market. Then we have*

$$CoER^= = \mu_p + \sigma_p \left(\rho\Phi^{-1}(q_m) - \frac{1}{q_p}\sqrt{1-\rho^2}\phi\left(\Phi^{-1}(q_p)\right) \right). \tag{3.5}$$

where $\phi(\cdot)$ ($\Phi(\cdot)$) is the standard normal pdf (cdf) and q_m (q_p) are the quantile levels in the defitinion of VaR (CoVaR), see (3.1) and (3.2).

Proof. See Appendix A. □

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Not surprisingly, $CoER^=$ is a function of the first two moments only, which is due to the assumption of jointly normal returns. The quantiles q_p and q_m that enter (3.5) can be thought of as model parameters. It should be noticed that

$$\frac{\mathrm{d}CoER^=}{\mathrm{d}\rho} = \sigma_p \left(\Phi^{-1}(q_m) + \frac{\rho}{q_p\sqrt{1-\rho^2}}\phi\left(\Phi^{-1}(q_p)\right) \right) \tag{3.6}$$

and $\frac{\mathrm{d}CoER^=}{\mathrm{d}\rho} < 0$ if, and only if,

$$\frac{\rho}{q_p\sqrt{1-\rho^2}} < -\frac{\Phi^{-1}(q_m)}{\phi\left(\Phi^{-1}(q_p)\right)} \tag{3.7}$$

which is not necessarily satisfied for values of ρ close to 1. In other words, for sufficiently high values of ρ the co-expected portfolio returns in stressed market conditions increase as the correlation with the market gets larger which is somewhat couterintuitive.

The optimal portfolio that maximizes $CoER^=$ is given in the following proposition.

Proposition 1. *Let w be a vector of weights of each asset in the portfolio, then under the assumptions of Lemma 1 we have*

$$CoER^= = w^\mathsf{T}\widehat{\mu} - \lambda\sqrt{w^\mathsf{T}\widehat{\Sigma}w} \tag{3.8}$$

where

$$\widehat{\mu} = \mu + \frac{\Phi^{-1}(q_m)}{\sigma_m}\sigma, \quad \lambda = \frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}, \quad \widehat{\Sigma} = \Sigma - \frac{1}{\sigma_m^2}\sigma\sigma^\mathsf{T} \tag{3.9}$$

with Σ being the covariance matrix of risky assets and σ being the column vector of covariances of each asset with the market index.

Furthermore, if

$$\lambda > \sqrt{\Delta^\mathsf{T}Q^{-1}\Delta} \tag{3.10}$$

the portfolio that maximizes $CoER^=$ is finite and is given by

$$w^* = \frac{\widehat{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\mathsf{T}\widehat{\Sigma}^{-1}\mathbf{1}} - \frac{1}{\sqrt{(\lambda^2 - \Delta^\mathsf{T}Q^{-1}\Delta)(\mathbf{1}^\mathsf{T}\widehat{\Sigma}^{-1}\mathbf{1})}} \begin{bmatrix} (Q^{-1})^\mathsf{T}\Delta \\ -\tilde{\mathbf{1}}^\mathsf{T}Q^{-1}\Delta \end{bmatrix} \tag{3.11}$$

where Q, $\tilde{\mathbf{1}}$, and Δ are given in Appendix B.

Proof. See Appendix B. □

To further analyse the results of Proposition 1 we note that $CoER^=$ can be written as

$$CoER^= = w^\mathsf{T}\mu + \frac{\Phi^{-1}(q_m)}{\sigma_m}w^\mathsf{T}\sigma - \frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}\sqrt{w^\mathsf{T}\Sigma w - \frac{1}{\sigma_m^2}\left(w^\mathsf{T}\sigma\right)^2} \qquad (3.12)$$

$$= \text{Portfolio's Return}$$

$$+ \frac{\Phi^{-1}(q_m)}{\sigma_m} \times \left(\text{Portfolio–Market covariance}\right)$$

$$- \frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p} \times \sqrt{\text{Portfolio's variance} - \frac{1}{\sigma_m^2}\left(\text{Portfolio–Market covariance}\right)^2}$$

which allows us the following interpretation of the model parameters.

First, the parameter q_m can be thought of as controlling the weight that investor assigns to Portfolio–Market covariance when maximizing portfolio's $CoER^=$. In this sense, small q_m implies large weight (large $\Phi^{-1}(q_m)$ in absolute terms) for the portfolio's covariance with the market. The underlying mechanism here is as follows. Relatively small q_m implies that we consider more adverse market return scenarios and, consequently, high correlation with the market becomes particularly dangerous for portfolio returns. This results in a larger penalty for higher (positive) correlation with the market in the objective function $CoER^=$.

Since $\Phi^{-1}(q_m)$ is negative for $q_m < 0.5$ (practically relevant case), investors who maximize $CoER^=$ of their portfolios will prefer portfolios with negative correlation with the market. In particular, when short-sales are not allowed, portfolio's negative correlation with the market implies that assets with negative covariance with the market (negative entries of σ) are more attractive. Intuitively, such portfolios should pay off well in crises when the market is in a dowturn.

Second, the parameter q_p can be thought of as controlling the weight that investors put on portfolio's variance. In this sense, small q_p implies large weight $\frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}$ assigned to the portfolio's variance (see Appendix C). Again, the mechanism that causes this behavior is the following. Small q_p implies that we consider more adverse portfolio returns. Since these returns are bounded from above by $CoVaR_{q_p}$, large portfolio variance implies large downside risk in the sense that more negative portfolio returns become more likely. As a consequence, the penalty for large portfolio variance increases.

Careful inspection of the $CoER^=$ representation (3.12) reveals that for fixed portfolio–market covariance (i.e., $w^\mathsf{T}\sigma = c_p$) and fixed expected portfolio returns (i.e., $w^\mathsf{T}\mu = \mu_p$), the

problem of finding the maximum of $CoER^=$ reduces to the problem of finding the minimum of the portfolio variance (i.e., $w^\mathsf{T}\Sigma w$). In this respect we have the following proposition.

Proposition 2. *Under the assumptions of Lemma 1, the optimal portfolio w^* given by (3.11) that maximizes $CoER^=$ belongs to the efficient boundary (σ_p, μ_p, c_p) that is defined as the solution to the following problem*

$$\min_w \quad \sigma_p^2 = w^\mathsf{T}\Sigma w \tag{3.13}$$

$$s.t. \quad w^\mathsf{T}\mathbf{1} = 1, \tag{3.14}$$

$$w^\mathsf{T}\mu = \mu_p, \tag{3.15}$$

$$w^\mathsf{T}\sigma = c_p, \tag{3.16}$$

where $[1, \mu_p, c_p] \neq [0, 0, 0]$. The efficient boundary (σ_p, μ_p, c_p) is given by the equation

$$\sigma_p^2 = [1, \mu_p, c_p] \left(B\Sigma^{-1}B^\mathsf{T} \right)^{-1} [1, \mu_p, c_p]^\mathsf{T} \tag{3.17}$$

where $B = [\mathbf{1}, \mu, \sigma]^\mathsf{T}$ is of full rank.

Furthermore, the following separation result holds: Any optimal portfolio with given expected return and covariance with the market can be replicated by three portfolios that belong to the efficient boundary.

Proof. See Appendix D. $\qquad\square$

Although the optimal portfolio w^* given in Proposition 1 (see Equation (3.11)) belongs to the boundary (σ_p, μ_p, c_p) specified by (3.17), there are points on the boundary that do not have portfolio represenations (3.11) (see Figure 2). Put it differently, there are specifications for the triple (σ_p, μ_p, c_p) that cannot be achieved by w^* specified by (3.11), that is, optimal portfolios obtained by varying the parameters q_m and q_p.

One should notice a similarity of $CoER^=$ in (3.8) with the mean–variance portfolio optimization. In this respect, we say that a portfolio w belongs to the mean–variance boundary if for some μ_p it solves

$$\min_w \quad \sigma_p^2 = w^\mathsf{T}\Sigma w \tag{3.18}$$

$$s.t. \quad w^\mathsf{T}\mathbf{1} = 1, \tag{3.19}$$

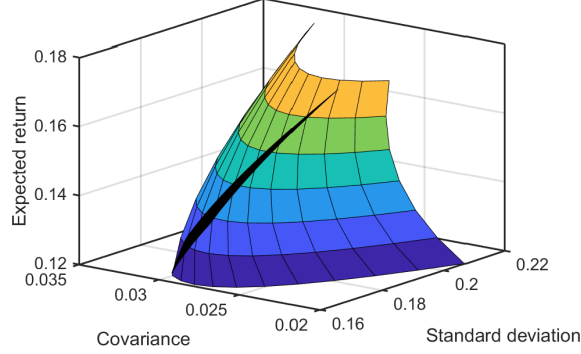$$w^\mathsf{T}\mu = \mu_p. \tag{3.20}$$

FIGURE 2. Comparison of the efficient boundaries. Boundary (σ_p, μ_p, c_p) (squares) that solves the problem (3.13) and the boundary (black patch) that solves the problem (**P1**) when $q_m = q_p = 0.1$.

Since the boundary (σ_p, μ_p, c_p) satisfies an additional constraint on the portfolio covariance with the market index (as compared with the mean–variance boundary), one should expect the mean–variance boundary to dominate (be above) the boundary (σ_p, μ_p, c_p) in (σ_p, μ_p)–space. In other words, the portfolios that are mean–variance efficient have higher expected returns for a given value of portfolio's standard deviation σ_p. In this respect we also have the following proposition.

Proposition 3. *If all assets in the portfolio are uncorrelated with the market, that is, if* $\sigma = \mathbf{0}$ *where* $\mathbf{0} = (0, ..., 0)^{\mathsf{T}} \in \mathbb{R}^n$, *then the portfolio* w^* *given by* (3.11) *that maximizes* $CoER^=$ *belongs to the mean–variance boundary.*

Proof. See Appendix E. □

From Proposition 3 it follows that when assets are uncorrelated with the market index, then the portfolio that maximizes $CoER^=$ is also mean–variance efficient. Next we analyze the case of $CoER^{\leq}$.

4. PORTFOLIO SELECTION: MARKET IS AT MOST AT ITS VAR LEVEL

In this section we take into account more severe market moves when selecting the optimal portfolio. To incorporate this assumption of more stressed market conditions, in Section 4.1 we modify the original definition of CoVaR of Adrian and Brinnermeier (2011) and in Section 4.2 we analyse the portfolio choice problem.

4.1. **Modified CoVaR.** Girardi and Ergün (2013) propose a modification of CoVaR where they condition at more extreme market downturns. In particular, in their modification they condition on $R_m \leq VaR_{q_m}$ instead of $R_m = VaR_{q_m}$, that is,

$$\mathbb{P}\left(R_p \leq CoVaR_{q_p} \middle| R_m \leq VaR_{q_m} \right) = q_p, \tag{4.1}$$

This definition allows for more severe losses (farther in the tail), i.e., those beyond VaR_{q_m}. Furthermore, this definition of CoVaR fixes the problem discussed in Section 3.2, namely, the fact that $CoES^{=}$ is not monotonic in the portfolio's correlation ρ with the market.

4.2. **Portfolio choice.** Based on the definition of CoVaR (4.1) we define the co-expected returns as

$$CoER^{\leq} = \mathbb{E}\left[R_p \middle| R_p \leq CoVaR_{q_p}, R_m \leq VaR_{q_m} \right]. \tag{4.2}$$

The returns on the market and the portfolio that we condition on in (4.2) are graphically illustrated in Figure 3.
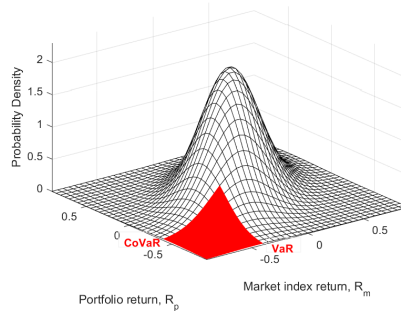

FIGURE 3. Illustration of the returns on which we condition in $CoER^{\leq}$. For a given probability distribution of returns on the market index and portfolio (in grey) we consider only those returns that correspond to stressed scenarios (in red): portfolio returns are below CoVaR ($R_p \leq CoVaR_{q_p}$) and market returns are at most at their VaR level ($R_m \leq VaR_{q_m}$).

Similarly to $CoES^{=}$, $CoER^{\leq}$ estimates the expected returns in a low return environment when the overall market is in distress (systemic risk index is at most at its VaR level). Thus,

the portfolio choice problem can be stated as

$$\max_{w} \quad CoER^{\leq} \tag{P2}$$

$$\text{s.t.} \quad w^{\mathsf{T}}\mathbf{1} = 1. \tag{4.3}$$

In other words, the optimal portfolio that solves (**P2**) is expected to perform well when the market is at most at its VaR level and portfolio's returns are below CoVaR.

In the following proposition we derive the closed-form expression for $CoER^{\leq}$.

Proposition 4. *Assume that* $(R_p, R_m) \sim \mathcal{BN}(\mu_p, \mu_m, \sigma_p, \sigma_m, \rho)$ *where* μ_p (μ_m) *is the expected portfolio (market) return,* σ_p (σ_m) *is portfolio (market) return standard deviation, and* $\rho \geq 0$ *is the correlation between the returns on the portfolio and the market. Then we have*

$$CoER^{\leq} = \mu_p - K(\rho; q_m, q_p)\sigma_p \tag{4.4}$$

where $K(\rho; q_m, q_p) > 0$ *and is given by*

$$K(\rho; q_m, q_p) = \frac{1}{q_m q_p} \left(\phi(\eta_1)\Phi\left(\frac{\eta_2 - \rho\eta_1}{\sqrt{1-\rho^2}}\right) + \rho\phi(\eta_2)\Phi\left(\frac{\eta_1 - \rho\eta_2}{\sqrt{1-\rho^2}}\right) \right). \tag{4.5}$$

with $\eta_1 = \frac{CoVaR_{q_p} - \mu_p}{\sigma_p}$, $\eta_2 = \frac{VaR_{q_m} - \mu_m}{\sigma_m}$, *and* $\phi(\cdot)$ ($\Phi(\cdot)$) *is the standard normal pdf (cdf). In addition, we have*

$$\frac{\mathrm{d}CoVaR_{q_p}}{\mathrm{d}\rho} < 0, \quad \frac{\mathrm{d}K}{\mathrm{d}\rho} > 0. \tag{4.6}$$

Proof. See Appendix F. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Unfortunately, there is no closed-form expression for the optimal portfolio weights w^* in problem (**P2**). Thus, we perform the analysys of this case numerically.

It follows from Proposition 4 that investors who use $CoER^{\leq}$ as a risk measure prefer portfolios with low standard deviation and low correlation with the market. If the function K were independent of ρ (implying that it is also independent of w), then the portfolio choice problem (**P2**) would have been equivalent to the classical mean–variance analysis (see also Proposition 5). Thus, function K adjusts the risk as measured only by the portfolio's standard deviation σ_p to account for the correlation between the portfolio and the market.

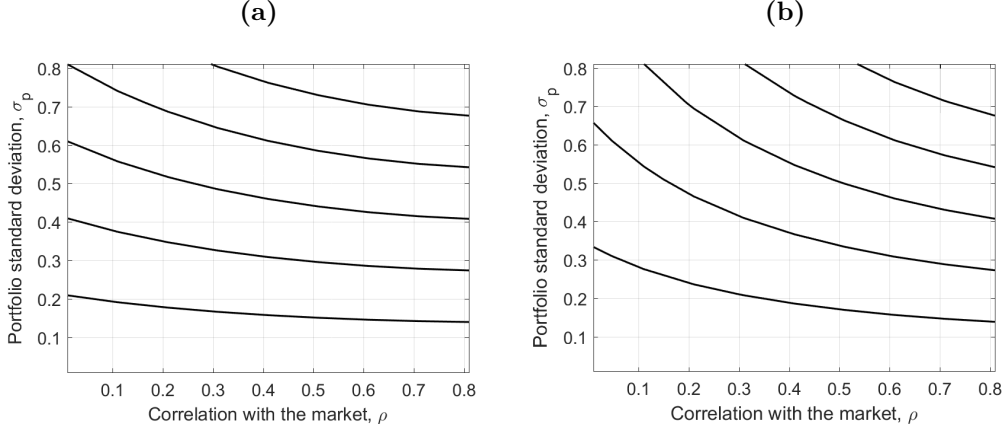|	(a)	(b)

FIGURE 4. Contour lines of $CoER^{\leq}$ for $\sigma_m = 0.9$ and $\mu_p = \mu_m = 0$: (a) $q_m = 10\%$ and $q_p = 10\%$, (b) $q_m = 10\%$ and $q_p = 50\%$

The contour lines for $CoER^{\leq}$ are shown in Figure 4.[4] Figure 4 reveals that the higher the standard deviation of portfolio returns, the larger role portfolio's correlation with the market plays in investor's portfolio choice. Indeed, as the standard deviation gets larger, the level curves become steeper. In this respect it should be noticed that the top-right corners of panels (a) and (b) in Figure 4 represent empirically relevant cases because correlation usually increases with volatility (see Longin and Solnik (1995), Campa and Chang (1998), Roll (1988), Black (1976)). In addition, Figure 3 (a) and (b) also supports the fact that as the parameter q_p decreases, portfolio's correlation ρ with the market becomes less significant for the investor because the standard deviation starts playing a larger role (see the discussion after Proposition 1).

Since both ρ and σ_p depend on the weights of each asset in the portfolio, by minimizing $CoER^{\leq}$ the investors choose a portfolio that strikes a proper balance between its correlation with the market and its standard deviation (for a given expected return). For example, let $q_m = q_p = 10\%$, $\sigma_m = 0.2$, $\mu_m = \mu_p = 0$, and consider two portfolios:

- Portfolio A with $\rho_A = 0.1$ and $\sigma_{p_A} = 0.4$ (low correlation/high variance portfolio)
- Portfolio B with $\rho_B = 0.9$ and $\sigma_{p_B} = 0.3$ (high correlation/low variance portfolio)

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[4]A very similar contour plot is obtained when the return distribution is modelled by Student t-distribution.

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For Portfolio A we have $CoER_A^{\leq} = -0.77$ and for Portfolio B we obtain $CoER_B^{\leq} = -0.80$ implying that, *ceteris paribus*, portfolio A is preferrable even though it has higher standard deviation. In other words, the portfolios that are optimal from the mean–variance perspective can actually be suboptimal when their correlation with the market is taken into account, or equivalently, when $CoER^{\leq}$ is used as a risk measure.

Obviously, when the market is in a downturn investors want the portfolios that have low correlation with the market and this is exactly what is captured by $CoER^{\leq}$. Indeed, by construction $CoER^{\leq}$ considers the portfolio returns that are less than $CoVaR_{q_p}$ in stressed market conditions. However, when the market is in a downturn, low correlation between the portfolio and the market implies higher value of $CoVaR_{q_p}$ (since $\frac{\mathrm{d}CoVaR_{q_p}}{\mathrm{d}\rho} < 0$) which, in turn, implies higher expected returns in stressed markets. Therefore, conditioned on a systemic event $\{R_m \leq VaR_{q_m}\}$, low correlation with the market implies higher portfolio's expected return and higher $CoER^{\leq}$.

Similarly to $CoER^{=}$ we have the following result.

Proposition 5. *If all assets in the portfolio are uncorrelated with the market, then the portfolio that maximizes $CoER^{\leq}$ belongs to the mean–variance boundary.*

Proof. See Appendix G. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Although it does not seem possible to obtain analytic expression for the optimal portfolio weights in problem (**P2**), Proposition 5 provides conditions when one can obtain the optimal portfolio weights. Indeed, Merton (1972) shows that a portfolio w belongs to the mean–variance boundary if, and only if,

$$\frac{\sigma_p^2}{1/C} - \frac{(\mu_p - A/C)^2}{D/C^2} = 1 \tag{4.7}$$

where $A = \mathbf{1}^{\mathsf{T}}\Sigma^{-1}\mu$, $\mathbf{1} = (1, ..., 1)^{\mathsf{T}} \in \mathbb{R}^n$, $B = \mu^{\mathsf{T}}\Sigma^{-1}\mu$, $C = \mathbf{1}^{\mathsf{T}}\Sigma^{-1}\mathbf{1}$, and $D = BC - A^2$. Therefore, from Proposition 5 it follows that any portfolio that satisfies (4.7) and that consists of assets uncorrelated with the market also maximizes $CoER^{\leq}$.

5. Results on Empirical Data

We now test our model on the data from the Canadian financial market. We first describe the estimation methodolgy and then apply it when finding optimal portfolios.

5.1. **Estimation methodology.** In this section we describe the methodology that will be used to implement and test the portfolio optimization problem (**P2**). Although the model described in Section 4 is a one-period model, we will use dynamic approaches to estimate distributions of returns. We explicitly indicate the time dependency of the variables by adding the subscript t, and for example, instead of r_i we will write $r_{i,t}$ to indicate the rate of return at time t.

5.1.1. *GARCH Dynamic Conditional Correlation (DCC) modelling.* Let the logarithmic returns be denoted as $\tilde{r}_t = (\tilde{r}_{1,t}, ..., \tilde{r}_{n+1,t})^\mathsf{T}$ where $\tilde{r}_{i,t} = \ln(1 + r_{i,t})$, $i = 1, ..., n+1$ and $(n+1)$st return is the return on the market index. We assume that conditional on the information set $\mathcal{F}_{t-1}$ available at time $t - 1$, the returns have an (unspecified) distribution $\mathcal{D}$ with zero mean and time-varying covariance

$$\tilde{r}_t \mid \mathcal{F}_{t-1} \sim \mathcal{D}(0, D_t C_t D_t) \tag{5.1}$$

where D_t^2 is the diagonal matrix with variances $\sigma_{i,t}^2$ of $\tilde{r}_{i,t}$ on the main diagonal.

We will use GJR–GARCH model of Glosten et al. (1993) to model the dynamics of variances, that is,

$$\sigma_{i,t}^2 = \omega_{Vi} + \alpha_{Vi}\tilde{r}_{i,t-1}^2 + \gamma_{Vi}\tilde{r}_{i,t-1}^2 I_{i,t-1} + \beta_{Vi}\sigma_{i,t-1}^2, \tag{5.2}$$

where

$$I_{i,t} = \begin{cases} 1, & \text{if } \tilde{r}_{i,t} < 0 \\ 0, & \text{otherwise,} \end{cases} \tag{5.3}$$

The correlation matrix is modelled for the volatility adjusted returns $\varepsilon_t = D_t^{-1}\tilde{r}_t$ by

$$C_t = \text{diag}(Q_t)^{-1/2} Q_t \, \text{diag}(Q_t)^{-1/2} \tag{5.4}$$

where Q_t is the pseudo-correlation matrix. The Dynamic Conditional Correlation (DCC) model then specifies the dynamics of the matrix Q_t as

$$Q_t = (1 - \alpha_C - \beta_C)\, C + \alpha_C \varepsilon_{t-1}\varepsilon_{t-1}^\mathsf{T} + \beta_C Q_{t-1}, \tag{5.5}$$

where C is the unconditional correlation matrix. The model is typically estimated by a two-step quasi-maximum likelihood estimation procedure (see Engle (2009)). We will refer to the above model specification as GARCH–DCC. The GARCH–DCC methodology is widely used in financial time series analysis as this class of models is parsimonious and is able to capture well many stylized facts of financial data.

5.1.2. *Future returns scenarios generation.* After the parameters of the GARCH–DCC model have been estimated, we can use them to obtain the distribution of future returns (see also Brownlees and Engle (2016)). In what follows we assume parameters to be known while in practice we use estimated parameters using all of the information available up to the current time T. Let h be the length of the time horizon for which the returns should be simulated. We generate future returns scenarios according to the following procedure.

(1) Obtain the standardized innovations

$$\epsilon_t = \left(U_t^\mathsf{T}\right)^{-1} \varepsilon_t, \ t = 1, ..., T, \tag{5.6}$$

   where U is the upper-triangular matrix in the Cholesky decomposition of the correlation matrix C_t.

(2) Sample with replacement $S \times h$ vactors of standardized innovations ϵ_t and use them to construct S pseudo samples of GARCH–DCC innovations over the horizon h, that is,

$$\epsilon_{T+t}^s, \ t = 1, ..., h, \ s = 1, ..., S. \tag{5.7}$$

(3) Use the pseudo samples from step 2 as inputs to GARCH–DCC model using as initial conditions the last values of the correlation matrix C_T and variances σ_{iT}^2. This step delivers S pseudo samples of GARCH–DCC logarithmic returns from time $T + 1$ to time $T + h$ conditional on the realized process up to time T, that is,

$$\tilde{r}_{T+t}^s, t = 1, ..., h \mid \mathcal{F}_T, \ s = 1, ..., S. \tag{5.8}$$

(4) Evaluate the multi-period return for each sample

$$r_{T+h}^s = \exp\left(\sum_{t=1}^h \tilde{r}_{T+h}^s\right) - 1. \tag{5.9}$$

The simulated future returns r_{T+h}^s are then used in the optimization procedure.

5.2. **Optimal portfolios.** In this section we empirically test our model. To generate the scenarious of future returns we use the GARCH–DCC model (see Engle (2002), (2009)) and apply the methodology to obtain ranking of Canadian financial institutions (five banks and three insurance companies) based on their systemic risk–return characteristics. In particular, we consider TD Bank, CIBC, RBC, Scotiabank, BMO, Manulife, Great West Lifeco, and Sunlife. Our choice of the Canadian financial services industry is primarily due

to its size as the banking and insurance sectors in Canada are represented by a relatively small number of companies and this, in turn, reduces the computational complexity of our estimation methodology (see Appendix 5.1). As a market index we use MSCI World Index (see also Brownlees and Engle (2016)). To estimate the parameters of the GARCH–DCC model we use Bloomberg data from January 3, 2000 until October 31, 2017. All time series are converted into Canadian dollars based on end-of-day exchange rates reported by Bloomberg. Table 1 provides the summary for model parameter estimates.

TABLE 1. The quantiles of GARCH–DCC parameter estimates based on the sample that spans January 3, 2006 to October 31, 2017 estimated for the firms in each category (banks, insurance companies) and the market index. The values are reported in form (10%,50%,90%)-quantiles.

	Banks	Insurance Companies	Market Index
ω_V	(0.0126, 0.0277, 0.0789)	(0.0179, 0.0290, 0.0428)	(0.0066, 0.0103, 0.0113)
α_V	(0.0340, 0.0613, 0.0780)	(0.0180, 0.0475, 0.0772)	(0.0000, 0.0017, 0.0065)
β_V	(0.8397, 0.8920, 0.9318)	(0.8748, 0.9205, 0.9330)	(0.9336, 0.9399, 0.9551)
γ_V	(0.0425, 0.0630, 0.1030)	(0.0364, 0.0676, 0.0880)	(0.0733, 0.0865, 0.0924)
α_C	(0.0074, 0.0080, 0.0095)	(0.0074, 0.0080, 0.0095)	(0.0074, 0.0080, 0.0095)
β_C	(0.9507, 0.9826, 0.9843)	(0.9507, 0.9826, 0.9843)	(0.9507, 0.9826, 0.9843)

The summary statistics on parameter estimates given in Table 1 reveals that the point estimates of the GJR–GARCH parameters are in line with the typical GJR–GARCH parameter estimates (see Engle (2002), (2009), Brownlees and Engle (2016)). Except for the intercept, DCC model parameter estimates are also close to the typical set of estimates and the parameters are similar across groups. One should also notice slightly different values of the asymmetric coefficient γ_V between financial institutions and the market index implying higher sensitivity of the market to large volatility increases in case of a drop in the index's value.

Next we backtest our model on daily data starting from January 3, 2001. We assume that our investor rebalances his/her portfolio every month starting from January 2007. Thus,

at the beginning of every month we generate $S = 500,000$ monthly return scenarios (see Section 5.1.2) and use these future scenarios to find the optimal portfolios. The failry large number of generated scenarios ensures that we have a good representation of extreme market events. As a benchmark portfolio we use the tangency portfolio, that is, the portfolio on the mean–variance boundary that has the highest expected return per unit risk (standard deviation). To avoid large negative portfolio positions during stressed market conditions, we preclude short sales. The portfolios' performance is shown in Figure 5.
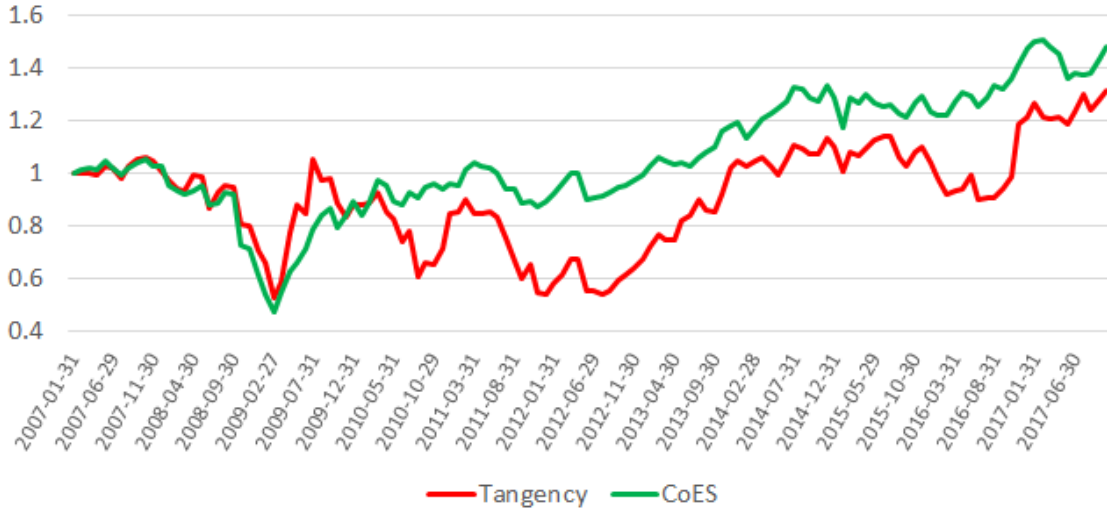

FIGURE 5. Out-of-sample performance of tangency and $CoER^{\leq}$ portfolios ($q_m = q_p = 0.1$) with monthly rebalancing and \$1 as an initial investment on January 31, 2007.

There are several notable features in Figure 5. First, both portfolios perform poorly during the 2007–2009 financial crisis and lose almost 50% of their value. The values of both portfolios significantly decline during this period because all stocks that we consider substantially lose in value. Second, and most importantly, $CoER^{\leq}$ portfolio has a superior performance during 2009–2012 European Sovereign Debt crisis when compared with the benchmark portfolio that loses almost 50%. Similarly, the benchmark portfolio value declines in the beginning of 2016 due to the declining price of oil, concerns regarding China's economic slowdown, and a weaker Canadian dollar. On the other hand, the $CoER^{\leq}$ portfolio value is fairly stable during this period. Third, the tangency portfolio has higher

volatility than the $CoER^{\leq}$ portfolio. In summary, the $CoER^{\leq}$ portfolio has an overall superior performance due to its relatively stable performance in periods of market downturns.

Next we look at ranking of financial institutions based on the relative proportions of their equities in the optimal portfolios. The institutions that have relatively high proportions in the optimal $CoER^{\leq}$ portfolio can be viewed as more attractive from "systemic risk–return" perspective. In this sense, this ranking is more valuable for investors than a more common, pure "systemic risk" ranking. For example, it is not clear what proportion of wealth investors should allocate to the institution that is ranked as, say, third according to some systemic risk ranking methodology. Figure 6 shows the portfolio constituents for the tangency and $CoER^{\leq}$ portfolios.
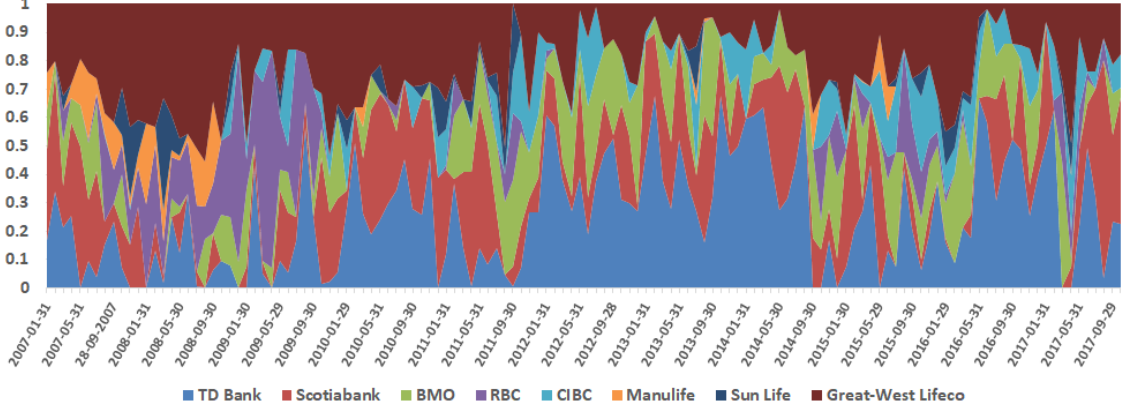
Importantly, from Figure 6 it follows that $CoER^{\leq}$ portfolio is more diversified than the tangency portfolio for which Manulife is the predominant equity starting from 2009. On the other hand, $CoER^{\leq}$ portfolio implies relatively high investment in TD Bank and Great-West Lifeco which have been well known to be among the most consistent performers.

6. Conclusions

In this paper we develop a model for portfolio choice in presence of systemic risk. Our modeling approach is based on two risk measures: VaR and CoVaR. The goal of the investor is to maximize the portfolio's expected returns conditioned on a systemic risk index being at (at most at) its VaR level and the portfolio's returns being below their CoVaR level.

Under some assumptions we derive the optimal portfolio in closed form. The parameters of the model allow the investor to properly balance portfolio's variance, expected return, and correlation with the systemic risk index. We show that any optimal portfolio with given expected return, variance, and covariance with the index can be replicated by three appropriately chosen portfolios. The optimal portfolio becomes mean–variance efficient when the assets are not correlated with the index. We apply our model to the Canadian equity market and by backtesting our methodology we find that at the times of market downturns our portfolio performs substantially better than the classical tangency portfolio that we use as a benchmark. In addition, we apply our model to ranking of financial institutions based on their share in the optimal portfolio and demonstrate that the most consistent performers have relatively large share in the optimal portfolio.

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(a) Composition of $CoER^{\leq}$ portfolio

(b) Composition of tangency portfolio
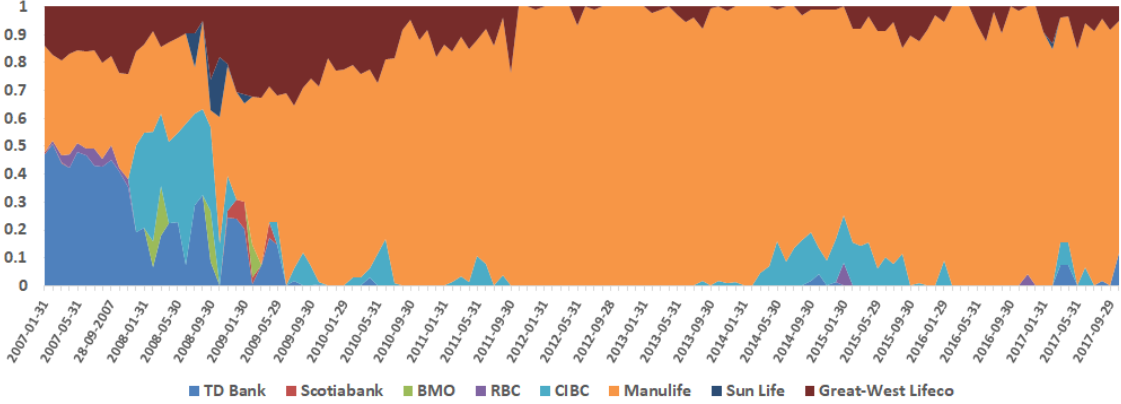

FIGURE 6. (a) $CoER^{\leq}$ and (b) tangency portfolios' composition.

APPENDIX A. PROOF OF PROPOSITION 1

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ then from the definition of VaR

$$\int_{-\infty}^{VaR_q} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \mathrm{d}x = q \tag{A.1}$$

we have

$$VaR_q = \mu + \sigma\Phi^{-1}(q). \tag{A.2}$$

The Expected Shortfall is given as

$$ES = \frac{1}{q} \int_{-\infty}^{VaR_q} \frac{x}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \mathrm{d}x$$

$$= \mu - \frac{\sigma}{q}\phi\left(\Phi^{-1}(q)\right). \tag{A.3}$$

Let $(X, Y) \sim \mathcal{BN}(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$. For fixed $Y = y$ it is well known that

$$X|Y = y \sim \mathcal{N}\left(\mu_x + \rho\sigma_x \frac{y - \mu_y}{\sigma_y}, \sigma_x^2(1-\rho^2)\right) \tag{A.4}$$

Let $R_p = X$ and $R_m = Y$. Since CoVaR is defined as the VaR of the portfolio given that the market is at its VaR level (i.e., we fix R_m at its VaR level $VaR_{q_m} = \mu_m + \sigma_m\Phi^{-1}(q_m)$) and applying (A.4) with (A.3), we have

$$CoER = \underbrace{\mu_p + \rho\sigma_p \frac{VaR_{q_m} - \mu_m}{\sigma_m}}_{\mu \text{ in (A.3)}} - \underbrace{\frac{\sigma_p}{q_p}\sqrt{1-\rho^2}}_{\frac{\sigma}{q} \text{ in (A.3)}} \phi\left(\Phi^{-1}(q_p)\right)$$

$$= \mu_p + \sigma_p\left(\rho\Phi^{-1}(q_m) - \frac{1}{q_p}\sqrt{1-\rho^2}\phi\left(\Phi^{-1}(q_p)\right)\right). \tag{A.5}$$

APPENDIX B. PROOF OF PROPOSITION 1

Let us write the covariance matrix of returns on all assets and the market index as

$$\begin{bmatrix} \Sigma & \sigma \\ \sigma^{\mathsf{T}} & \sigma_m^2 \end{bmatrix} \tag{B.1}$$

where σ is the column vector of covariances of each asset with the market index. The joint distribution of portfolio's return and the market index return is bivariate normal with the mean vector $[w^{\mathsf{T}}\mu, \mu_m]$ and the covariance matrix

$$\begin{bmatrix} w^{\mathsf{T}}\Sigma w & w^{\mathsf{T}}\sigma \\ w^{\mathsf{T}}\sigma & \sigma_m^2 \end{bmatrix} = \begin{bmatrix} \sigma_p^2 & w^{\mathsf{T}}\sigma \\ w^{\mathsf{T}}\sigma & \sigma_m^2 \end{bmatrix}. \tag{B.2}$$

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Since the portfolio correlation with the market is $\rho = w^\mathsf{T}\sigma/\sigma_m\sigma_p$, we have from (A.5) that

$$CoER^= = \mu_p + \sigma_p\left(\rho\Phi^{-1}(q_m) - \frac{1}{q_p}\sqrt{1-\rho^2}\phi\left(\Phi^{-1}(q_p)\right)\right)$$

$$= w^\mathsf{T}\mu + \sqrt{w^\mathsf{T}\Sigma w}\left(\frac{w^\mathsf{T}\sigma}{\sigma_m\sqrt{w^\mathsf{T}\Sigma w}}\Phi^{-1}(q_m) - \frac{1}{q_p}\sqrt{1-\left(\frac{w^\mathsf{T}\sigma}{\sigma_m\sqrt{w^\mathsf{T}\Sigma w}}\right)^2}\phi\left(\Phi^{-1}(q_p)\right)\right)$$

$$= w^\mathsf{T}\mu + \frac{\Phi^{-1}(q_m)}{\sigma_m}w^\mathsf{T}\sigma - \frac{\phi\left(\Phi^{-1}(q_p)\right)}{\sigma_m q_p}\sqrt{\sigma_m^2 w^\mathsf{T}\Sigma w - w^\mathsf{T}\sigma\sigma^\mathsf{T}w}$$

$$= w^\mathsf{T}\left(\mu + \frac{\Phi^{-1}(q_m)}{\sigma_m}\sigma\right) - \frac{\phi\left(\Phi^{-1}(q_p)\right)}{\sigma_m q_p}\sqrt{\sigma_m^2 w^\mathsf{T}\left(\Sigma - \frac{1}{\sigma_m^2}\sigma\sigma^\mathsf{T}\right)w}$$

$$= w^\mathsf{T}\widehat{\mu} - \lambda\sqrt{w^\mathsf{T}\widehat{\Sigma}w} \tag{B.3}$$

where

$$\widehat{\mu} = \mu + \frac{\Phi^{-1}(q_m)}{\sigma_m}\sigma, \quad \lambda = \frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}, \quad \widehat{\Sigma} = \Sigma - \frac{1}{\sigma_m^2}\sigma\sigma^\mathsf{T}. \tag{B.4}$$

From Landsman (2008,a) it follows that the maximum of (B.3) under the constraint $w^\mathsf{T}\mathbf{1} = 1$ is reached at

$$w^* = \frac{\widehat{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\mathsf{T}\widehat{\Sigma}^{-1}\mathbf{1}} - \frac{1}{\sqrt{(\lambda^2 - \Delta^\mathsf{T}Q^{-1}\Delta)(\mathbf{1}^\mathsf{T}\widehat{\Sigma}^{-1}\mathbf{1})}}\begin{bmatrix} (Q^{-1})^\mathsf{T}\Delta \\ -\tilde{\mathbf{1}}^\mathsf{T}Q^{-1}\Delta \end{bmatrix} \tag{B.5}$$

where $\Delta = \widehat{\mu}_n\tilde{\mathbf{1}} - \tilde{\widehat{\mu}}$, $\tilde{\widehat{\mu}} = [\widehat{\mu}_1, ..., \widehat{\mu}_{n-1}]^\mathsf{T}$, $\tilde{\mathbf{1}} = (1, ..., 1)^\mathsf{T} \in \mathbb{R}^{n-1}$, $Q = \tilde{\Sigma} - \tilde{\mathbf{1}}\tilde{\sigma}^\mathsf{T} - \tilde{\sigma}\tilde{\mathbf{1}}^\mathsf{T} + \sigma_{nn}\tilde{\mathbf{1}}\tilde{\mathbf{1}}^\mathsf{T}$, and

$$\widehat{\Sigma} = \begin{bmatrix} \tilde{\Sigma} & \tilde{\sigma} \\ \tilde{\sigma}^\mathsf{T} & \sigma_{nn} \end{bmatrix} \tag{B.6}$$

privided that

$$\lambda > \sqrt{\Delta^\mathsf{T}Q^{-1}\Delta}. \tag{B.7}$$

The restriction (B.7) is to ensure a finite solution.

Appendix C. Proof of $\frac{\mathrm{d}\lambda}{\mathrm{d}q_p} < 0$

To show that $\frac{\mathrm{d}\lambda}{\mathrm{d}q_p}$ is negative, we evaluate

$$\frac{\mathrm{d}\lambda}{\mathrm{d}q_p} = \frac{\mathrm{d}}{\mathrm{d}q_p}\left(\frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}\right) = -\frac{q_p\Phi^{-1}(q_p) + \phi\left(\Phi^{-1}(q_p)\right)}{q_p^2} \tag{C.1}$$

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where we used the inverse function theorem. Clearly, $\frac{\mathrm{d}}{\mathrm{d}q_p}\left(\frac{\phi\left(\Phi^{-1}(q_p)\right)}{q_p}\right) < 0$ is equivalent to

$$q_p\Phi^{-1}(q_p) + \phi\left(\Phi^{-1}(q_p)\right) > 0. \tag{C.2}$$

Let $x = \Phi^{-1}(q_p)$, then inequality (C.2) becomes $f(x) := x\Phi(x) + \phi(x) > 0$ which is true because $\frac{\mathrm{d}f}{\mathrm{d}x} = \Phi(x) > 0$ and $\lim_{x \to -\infty} f(x) = 0$ where we applied L'Hospital's rule twice to the fraction $\frac{x}{1/\Phi(x)}$.

Appendix D. Proof of Proposition 2

From Landsman (2008,b) we have the following solution to the constrained quadratic problem

$$w^{**} = \Sigma^{-1}B^{\mathsf{T}}\left(B\Sigma^{-1}B^{\mathsf{T}}\right)^{-1}[1, \mu_p, c_p]^{\mathsf{T}} \tag{D.1}$$

which yields the efficient boundary

$$\sigma_p^2 = (w^{**})^{\mathsf{T}}\Sigma w^{**} = [1, \mu_p, c_p]\left(B\Sigma^{-1}B^{\mathsf{T}}\right)^{-1}[1, \mu_p, c_p]^{\mathsf{T}}. \tag{D.2}$$

To see that the optimal portfolio w^* (see (3.11)) belongs to the efficient boundary one should substitute $(w^*)^{\mathsf{T}}\Sigma w^*$, $(w^*)^{\mathsf{T}}\mu$, $(w^*)^{\mathsf{T}}\sigma$, for σ_p^2, μ_p, c_p, respectively, in (D.2). With this substitution, the equality holds.

Next, we establish the separation result. Consider the portfolios specified by the following vectors $p_i = [1, \mu_{p,i}, c_{p,i}]^{\mathsf{T}}$, $i = 1, 2, ..., N_p$ and assume that we want to replicate a portfolio specified by $\bar{p} = [1, \bar{\mu}_p, \bar{c}_p]$. Let α_i be the proportion of wealth invested in each portfolio, then applying (D.1) we have

$$\sum_{i=1}^{N_p}\alpha_i w_i^{**} = \Sigma^{-1}B^{\mathsf{T}}\left(B\Sigma^{-1}B^{\mathsf{T}}\right)^{-1}\left(\sum_{i=1}^{N_p}\alpha_i p_i\right) \tag{D.3}$$

which is equal to

$$\Sigma^{-1}B^{\mathsf{T}}\left(B\Sigma^{-1}B^{\mathsf{T}}\right)^{-1}[1, \bar{\mu}_p, \bar{c}_p]^{\mathsf{T}} \tag{D.4}$$

if and only if

$$\sum_{i=1}^{N_p}\alpha_i p_i = \bar{p}. \tag{D.5}$$

From the system (D.5) it is clear that, in general, three protfolios ($N_p = 3$) are required to guarantee the replication provided that $\det\left([p_1, p_2, p_3]\right) \neq 0$.

Appendix E. Proof of Proposition 3

If $\sigma = \mathbf{0}$, then from (3.9) we have that $\widehat{\mu} = \mu$ and $\widehat{\Sigma} = \Sigma$. Merton (1972) shows that a portfolio w belongs to the mean-variance boundary if and only if

$$\frac{\sigma_p^2}{1/C} - \frac{(\mu_p - A/C)^2}{D/C^2} = 1 \tag{E.1}$$

where $A = \mathbf{1}^\mathsf{T}\Sigma^{-1}\mu$, $\mathbf{1} = (1, ..., 1)^\mathsf{T} \in \mathbb{R}^n$, $B = \mu^\mathsf{T}\Sigma^{-1}\mu$, $C = \mathbf{1}^\mathsf{T}\Sigma^{-1}\mathbf{1}$, and $D = BC - A^2$. It is straightforward to verify that w^* in (3.11) solves (E.1).

Appendix F. Proof of Proposition 4

We use the results from Kan and Robotti (2016). On page 17, for $(X_1, X_2) \sim \mathcal{BN}(m, \Sigma)$ where Σ is the correlation matrix ($\sigma_{11} = \sigma_{22} = 1$, $\sigma_{12} = \sigma_{21} = \rho$), the authors provide the expression for the first moment of the *lower* truncated random variable Z_1, that is,

$$\mathbb{E}[Z_1] := \mathbb{E}[X_1 | a_1 < X_1, a_2 < X_2] = m_1 + \frac{\phi(\eta_1)\Phi(w_{2\cdot1}) + \rho\phi(\eta_2)\Phi(w_{1\cdot2})}{\Phi_2(\eta_1, \eta_2; \rho)} \tag{F.1}$$

where

$$\eta_i = m_i - a_i, \tag{F.2}$$

$$\Phi_2(\eta_1, \eta_2; \rho) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{\eta_1} \int_{-\infty}^{\eta_2} \exp\left(-\frac{(x^2 + y^2 - 2\rho xy)}{2(1 - \rho^2)}\right) \mathrm{d}x\,\mathrm{d}y, \tag{F.3}$$

$$w_{i\cdot j} = (\eta_i - \rho\eta_j)/\sqrt{1 - \rho^2}. \tag{F.4}$$

The result for general Σ can be obtained by replacing a_i with a_i/σ_i, m_i with m_i/σ_i, and multiplying the result by σ_1.

To obtain the first moment of the *upper* truncated bivariate normal random variable one should notice that $(-X_1, -X_2) \sim \mathcal{BN}(-m, \Sigma)$, and thus, replace m with $-m$, a with $-b$, and multiply the result by -1. In other words, we have

$$\mathbb{E}[X_1 | X_1 < b_1, X_2 < b_2] = m_1 - \sigma_1\frac{\phi(\eta_1)\Phi(w_{2\cdot1}) + \rho\phi(\eta_2)\Phi(w_{1\cdot2})}{\Phi_2(\eta_1, \eta_2; \rho)} \tag{F.5}$$

where $\eta_i = (b_i - m_i)/\sigma_i$. Since in our case $\Phi_2(\eta_1, \eta_2; \rho) = q_m q_p$, we obtain the result (4.4).

Since normal cdf and pdf are positive functions and $\rho \geq 0$, we also have $K(\rho; q_m q_p) > 0$. Furthermore, from the definition of $CoVaR_{q_p}$ we have the following

$$\mathbb{P}\left(R_p \leq CoVaR_{q_p} \middle| R_m \leq VaR_{q_m}\right) = q_p \tag{F.6}$$

$$\Longleftrightarrow \mathbb{P}\left(\frac{R_p - \mu_p}{\sigma_p} \leq \frac{CoVaR_{q_p} - \mu_p}{\sigma_p} \middle| \frac{R_m - \mu_m}{\sigma_m} \leq \frac{VaR_{q_m} - \mu_m}{\sigma_m}\right) = q_p \tag{F.7}$$

$$\Longleftrightarrow \mathbb{P}\left(n_1 \leq \eta_1 \middle| n_2 \leq \eta_2\right) = q_p, \tag{F.8}$$

where (n_1, n_2) have a bivariate standard normal distribution. Therefore, η_1 and η_2 are independent of μ_p, μ_m, σ_m, and σ_p implying that $K(\rho; q_m, q_p)$ is also independent of these parameters.

Next we show that function K is strictly increasing in correlation ρ. For compactness of notation in this appendix we will use X and Y instead of R_p and R_m, respectively. Thus, let $(X, Y) \sim \mathcal{BN}(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$, and define $\xi_1 := (x - \mu_x)/\sigma_x$, and $\xi_2 := (y - \mu_y)/\sigma_y$. As is well known, a bivariate normal pdf can be written as

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{(\xi_1^2 + \xi_2^2 - 2\rho\xi_1\xi_2)}{2(1-\rho^2)}\right) = \frac{1}{\sigma_x\sigma_y\sqrt{1-\rho^2}}\phi\left(\frac{\xi_1 - \rho\xi_2}{\sqrt{1-\rho^2}}\right)\phi(\xi_2) \tag{F.9}$$

$$= \frac{1}{\sigma_x\sigma_y\sqrt{1-\rho^2}}\phi\left(\frac{\xi_2 - \rho\xi_1}{\sqrt{1-\rho^2}}\right)\phi(\xi_1). \tag{F.10}$$

Therefore, the bivariate normal cdf becomes

$$\mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^{a}\int_{-\infty}^{b}\frac{1}{\sigma_x\sigma_y\sqrt{1-\rho^2}}\phi\left(\frac{\xi_2 - \rho\xi_1}{\sqrt{1-\rho^2}}\right)\phi(\xi_1)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_{-\infty}^{\frac{a-\mu_x}{\sigma_x}}\int_{-\infty}^{\frac{b-\mu_y}{\sigma_y}}\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{\xi_2 - \rho\xi_1}{\sqrt{1-\rho^2}}\right)\phi(\xi_1)\,\mathrm{d}\xi_2\,\mathrm{d}\xi_1$$

$$= \int_{-\infty}^{\frac{a-\mu_x}{\sigma_x}}\frac{1}{\sqrt{1-\rho^2}}\phi(\xi_1)\int_{-\infty}^{\frac{b-\mu_y}{\sigma_y}}\phi\left(\frac{\xi_2 - \rho\xi_1}{\sqrt{1-\rho^2}}\right)\mathrm{d}\xi_2\,\mathrm{d}\xi_1$$

$$= \int_{-\infty}^{\frac{a-\mu_x}{\sigma_x}}\phi(\xi_1)\int_{-\infty}^{\frac{\frac{b-\mu_y}{\sigma_y}-\rho\xi_1}{\sqrt{1-\rho^2}}}\phi(z)\,\mathrm{d}z\,\mathrm{d}\xi_1$$

$$= \int_{-\infty}^{\frac{a-\mu_x}{\sigma_x}}\Phi\left(\frac{\frac{b-\mu_y}{\sigma_y} - \rho\xi_1}{\sqrt{1-\rho^2}}\right)\phi(\xi_1)\,\mathrm{d}\xi_1 \tag{F.11}$$

It is also well known (see, for example, the Appendix in Sibuya (1960)) that

$$\frac{\mathrm{d}}{\mathrm{d}\rho} \int_{-\infty}^{c_1} \Phi\left(\frac{c_2 - \rho z}{\sqrt{1 - \rho^2}}\right) \phi(z) \, \mathrm{d}z = \phi_2(c_1, c_2; \rho) \tag{F.12}$$

where c_1 and c_2 are given constants and ϕ_2 denotes the pdf of a standard bivariate normal distribution (see also (F.3) for the cdf of a standard bivariate normal distribution). The result (F.12) will be used later.

To make our notation concise, let $a = CoVaR_{q_p}$ and $b = VaR_{q_m}$, then from (F.11), the definition of $CoVaR_{q_p}$ (see (4.1)), and the definition of VaR_{q_m} (see (3.1)) we have

$$\mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{\sigma_x \sigma_y \sqrt{1 - \rho^2}} \phi\left(\frac{\xi_2 - \rho \xi_1}{\sqrt{1 - \rho^2}}\right) \phi(\xi_1) \, \mathrm{d}y \, \mathrm{d}x = q_m q_p$$

$$\iff \int_{-\infty}^{\frac{a - \mu_x}{\sigma_x}} \Phi\left(\frac{\frac{b - \mu_y}{\sigma_y} - \rho \xi_1}{\sqrt{1 - \rho^2}}\right) \phi(\xi_1) \, \mathrm{d}\xi_1 = q_m q_p$$

$$\iff \int_{-\infty}^{\eta_1(\rho)} \Phi\left(\frac{\eta_2 - \rho \xi_1}{\sqrt{1 - \rho^2}}\right) \phi(\xi_1) \, \mathrm{d}\xi_1 = q_m q_p \tag{F.13}$$

where we used the definition of $\eta_1 = (a - \mu_x)/\sigma_x$ and $\eta_2 = (b - \mu_y)/\sigma_y$. Notice that η_2 is independent of ρ because marginal distributions of a bivariate normal distribution are independent of the correlation parameter and it is only the marginal distribution of Y that is used to evaluate VaR_{q_m}.

Implicit differentiation of (F.13) with respect to ρ yields

$$\Phi\left(\frac{\eta_2 - \rho \eta_1(\rho)}{\sqrt{1 - \rho^2}}\right) \phi(\eta_1(\rho)) \frac{\mathrm{d}\eta_1}{\mathrm{d}\rho} + \underbrace{\int_{-\infty}^{\eta_1(\rho)} \frac{\mathrm{d}}{\mathrm{d}\rho} \Phi\left(\frac{\eta_2 - \rho \xi_1}{\sqrt{1 - \rho^2}}\right) \phi(\xi_1) \, \mathrm{d}\xi_1}_{=\phi_2(\eta_1, \eta_2; \rho)} = 0 \tag{F.14}$$

where we applied (F.12). Thus, using (F.10) we obtain

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho} = -\frac{\frac{1}{\sqrt{1 - \rho^2}} \phi\left(\frac{\eta_2 - \rho \eta_1}{\sqrt{1 - \rho^2}}\right)}{\Phi\left(\frac{\eta_2 - \rho \eta_1}{\sqrt{1 - \rho^2}}\right)}. \tag{F.15}$$

In addition, from the definition of $CoVaR_{q_p}$ and from (F.15) we have that $\frac{\mathrm{d}CoVaR_{q_p}}{\mathrm{d}\rho} < 0$.

Finally, we differentiate $K(\rho; q_m, q_p)$ given by (4.5) with respect to ρ. First we evaluate

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\eta_2 - \rho\eta_1}{\sqrt{1-\rho^2}}\right) = \frac{\rho\left(\eta_2 - (1-\rho^2)\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho}\right) - \eta_1}{(1-\rho^2)^{3/2}}, \tag{F.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\eta_1 - \rho\eta_2}{\sqrt{1-\rho^2}}\right) = -\frac{\eta_2 - (1-\rho^2)\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho} - \rho\eta_1}{(1-\rho^2)^{3/2}}, \tag{F.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) = -x\phi(x). \tag{F.18}$$

Differentiation of the first term in the brackets of $K(\rho; q_m, q_p)$ (see its definition (4.5)) yields

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\phi(\eta_1)\Phi\left(\frac{\eta_2 - \rho\eta_1}{\sqrt{1-\rho^2}}\right)\right) = \underbrace{\frac{\rho}{\sqrt{1-\rho^2}}\phi(\eta_1)\phi\left(\frac{\eta_2 - \rho\eta_1}{\sqrt{1-\rho^2}}\right)}_{=\rho\phi_2(\eta_1;\eta_2;\rho)}\left(\frac{\eta_2 - \rho\eta_1}{1-\rho^2} - \frac{\mathrm{d}\eta_1}{\mathrm{d}\rho}\right) \tag{F.19}$$

where we used (F.15). For the second term in the brackets of $K(\rho; q_m, q_p)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi(\eta_2)\Phi\left(\frac{\eta_1 - \rho\eta_2}{\sqrt{1-\rho^2}}\right)\right) = \phi(\eta_2)\Phi\left(\frac{\eta_1 - \rho\eta_2}{\sqrt{1-\rho^2}}\right) \tag{F.20}$$

$$+ \underbrace{\frac{\rho}{\sqrt{1-\rho^2}}\phi(\eta_2)\phi\left(\frac{\eta_1 - \rho\eta_2}{\sqrt{1-\rho^2}}\right)}_{=\rho\phi_2(\eta_1;\eta_2;\rho)}\left(\frac{\mathrm{d}\eta_1}{\mathrm{d}\rho} - \frac{\eta_2 - \rho\eta_1}{1-\rho^2}\right). \tag{F.21}$$

Thus, we have

$$\frac{\mathrm{d}K}{\mathrm{d}\rho} = \frac{1}{q^2}\phi(\eta_2)\Phi\left(\frac{\eta_1 - \rho\eta_2}{\sqrt{1-\rho^2}}\right) > 0. \tag{F.22}$$

APPENDIX G. PROOF OF PROPOSITION 5

Let us write the covariance matrix of returns on all assets and the market index as

$$\begin{bmatrix} \Sigma & \sigma \\ \sigma^\mathsf{T} & \sigma_m^2 \end{bmatrix} \tag{G.1}$$

where σ is the column vector of covariances of each asset with the market index. The joint distribution of portfolio's return and the market index return is bivariate normal with the mean vector $[w^\mathsf{T}\mu, \mu_m]$ and the covariance matrix

$$\begin{bmatrix} w^\mathsf{T}\Sigma w & w^\mathsf{T}\sigma \\ w^\mathsf{T}\sigma & \sigma_m^2 \end{bmatrix} = \begin{bmatrix} \sigma_p^2 & w^\mathsf{T}\sigma \\ w^\mathsf{T}\sigma & \sigma_m^2 \end{bmatrix}. \tag{G.2}$$

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Since $\rho = w^\mathsf{T}\sigma/\sigma_m\sigma_p$, we have that if the assets are uncorrelated with the market index then $\rho = 0$. Thus, it follows that $K(\rho; q_m, q_p) = K(0; q_m, q_p)$ and it is independent of the portfolio weights w. Therefore, subject to the constraint $w^\mathsf{T}\mu = \mu_p$ we have

$$\max_w CoER^{\leq} = \min_w K(0; q_m, q_p)\sigma_p = K(0; q_m, q_p)\min_w \sigma_p. \tag{G.3}$$

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